POLYNOMIALS ON A FINITE DISCRETE RANGE

By E. T. Bell

A function is said to be on a range, to a range, if the values of the function are in the range for values of the variables in the range. An elementary algebra A_n of the system of polynomials on the range $R_n \equiv 0, \cdots, n-1$ to R_n is of interest on its own account, and may be of some use for its possible applications. For n fixed, a subsystem $A_n^{(2)}$ of A_n is abstractly identical (simply isomorphic) with the sentential (or propositional) calculus S_n for an n-valued truth system. (The few technical terms used in the sequel are explained in recent American texts on algebra and logic. For algebra, see [1], [4], [2]. For logic, see [3; Chapter 7], [5], [6]. The technical details of modern algebra are not required here; some of them, however, come into use at a later stage of the natural development, that of A_n as an iterative algebra over a commutative semi-group, or of A_n as a special type of modular system.)

Instead of polynomials on R_n to R_n, polynomials on $U_n \equiv u_0, \cdots, u_{n-1}$ to $V_n \equiv v_0, \cdots, v_{n-1}$ may be discussed essentially as for A_n (R_n to R_n). The formulas are more complicated, and there is no gain in generality, since the resulting algebra is isomorphic to A_n. However, since S_n is sometimes represented on and to the range $j/(n-1)$, $j = 0, \cdots, n-1$, the isomorphism between A_n and the algebra of polynomials on V_n to V_n will be noted (§9).

For any integer $s > 1$, a subsystem $A_n^{(s)}$ of A_n is closed under composition of n^{n^s} s-ary operations. In $S_2^{(2)}$, the sentential isomorph of $A_2^{(2)}$, a small set of the 16 possible operations is distinguished for its logical significance; these usually are negation, $(\ )'$, representable as a binary operation, conjunction, $\wedge$, disjunction, $\vee$, implication, $\rightarrow$, formal equivalence or the biconditional, $\equiv$, and rejection, $|$. Similarly for $S_3^{(2)}$. In $A_n^{(s)}$ no operations (for any s) are singled out for special attention over the others, as distinctions are of importance only in particular interpretations of the abstract system. Only experience will show what operations should be given precedence in $S_n^{(2)}$, $n > 2$; and it seems unlikely, to extrapolate from the long historical development of $S_2^{(2)}$, that such experience will come out of mathematics alone. Until experience materializes, the mere algebra of A_n (or of S_n) should provide numerous interesting exercises in imagination and manipulative skill.

As one of its applications, the algebra of A_n reduces the construction of systems having specified structural characteristics, also the generalization of these characteristics, such as commutativity with respect to one operation, transitivity and non-associativity with respect to another, and so on, to a non-tentative algorithm concerning multilinear Diophantine equations in which the values of the indeterminates are restricted to a finite range of non-negative integers.

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