DIFFERENTIALS AND ANALYTIC CONTINUATION IN NON-COMMUTATIVE ALGEBRAS

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This paper represents an attempt to generalize the power series portion of the theory of functions of a complex variable so that the argument may range over a non-commutative algebra.

For this purpose, an absolute value, or metric, is defined for the algebra. This absolute value is used to discuss the convergence of generalized power series. By introducing differentials, a generalization of Taylor's theorem is obtained. This involves several theorems concerning the existence and differentiability of generalized power series. The final section of the paper consists of the application of this theory to the special case of functions defined on the complete matrix algebra.

One of the important differences between this generalization and the classical theory is that the region of convergence and the region of absolute convergence are very different when a general algebra is considered. This complicates the theory of convergence of series, but is a blessing in the analytic continuation of the function.

German letters will be used to denote sets of elements; capital letters denote elements of the algebra; and small letters will denote elements of the field used to build the algebra.

1. **The metric.** Let $\mathfrak{A}$ be a linear associative algebra over the field of all real numbers $\mathfrak{R}$, or the field of all complex numbers $\mathfrak{C}$. These fields are chosen because they are complete and the proof of the existence of limits is greatly simplified. Besides, each of these fields has a commonly used absolute value which can be extended to $\mathfrak{A}$.

An absolute value for $\mathfrak{A}$ should have the properties:

$$|X| \geq 0; \quad |X| = 0 \text{ if, and only if, } X = 0; \tag{1.1}$$

$$|X + Y| \leq |X| + |Y|; \tag{1.2}$$

$$|XY| \leq |X|\;|Y|; \tag{1.3}$$

$$|xY| = |x|\;|Y|; \tag{1.4}$$

$$|I| = 1, \text{ if } \mathfrak{A} \text{ has an identity}; \tag{1.5}$$

$$|X| \text{ is continuous in coördinates for basis of } \mathfrak{A}. \tag{1.6}$$

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