FUNCTIONS OF BOUNDED TYPE

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Introduction. By definition, a function $f(z)$ is of bounded type (beschränktartig [5]) in a region G, if

$$(1) \qquad f(z) = W_2(z)/W_1(z) \qquad \text{in } G,$$

where $|W_i(z)| \leq 1$ and $W_i(z)$ are analytic interior to G $(i = 1, 2)$.

A given meromorphic function $f(z)$ is of bounded type in a simply connected region G if and only if there exists a function $U_1(z)$, positive and harmonic in G and such that

$$(2) \qquad \log |f(z)| \leq U_1(z) \qquad \text{in } G.$$

(Throughout this paper we shall use "harmonic" to describe functions which are harmonic in the strict sense except at isolated points b, where they behave like $\pm k \log |z - b|$.) For, if (1) holds, then so must (2) with $U_1 = \log |1/W_1|$, whereas, if (2) holds, then $\log |f(z)| = U_1(z) - U_2(z)$, where $U_2(z)$ is also positive and harmonic in G. The fact that one can obtain from $U_i(z)$ analytic functions $W_i(z)$ such that $U_i = \log |1/W_i|$ is discussed by R. Nevanlinna [5].

Our problem is to determine under what conditions $f(z)$, meromorphic in the interior of a region G, and perhaps on part of its boundary, is of bounded type in G, and, if $f(z)$ is of bounded type, to find a representation for $\log |f(z)|$. (The reader will notice that in the theorems of this paper $\log |f(z)|$ may be replaced by a harmonic function $U(z)$. The theorems then give conditions under which $U(z)$ may be expressed as the difference between two positive harmonic functions in G.) Explicit solutions of this problem have been given in two instances by R. Nevanlinna. The first solution [4] is for the case in which G is the halfplane, $x > 0$, $f(z)$ being meromorphic when $x \geq 0$; the second [5] is for the case in which G is the unit circle and $f(z)$ is meromorphic interior to G. The case in which G is a strip, $a < x < b$, has been considered by E. Hille [3] in a closely related problem, and by L. V. Ahlfors [1] in a theorem the proof of which demands more restrictive conditions on $f(z)$ than those which we shall use. The conditions found by these writers are of two kinds: (1) a certain sum extended over the poles of $f(z)$ in G must be convergent; (2) a mean value of the function, expressed as a weighted integral, on curves which approach the boundary of G, must remain finite. The reader will find conditions of both types in the theorems below.

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