QUASI-ANALYTICITY AND PROPERTIES OF FLATNESS OF ENTIRE FUNCTIONS

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Introduction. We shall denote by $C\{M_n\}$ the class of all infinitely differentiable functions which are defined in a closed interval $[a, b]$ and which have the following property: To each function $f(x)$ of the class there corresponds a constant $k = k(f)$ such that $|f^{(n)}(x)| < k^n M_n$ $(n \geq 1, x \in [a, b])$.

The class $C\{M_n\}$ is said to be quasi-analytic if from the two properties $f(x) \in C\{M_n\}, f^{(n)}(x_0) = 0$ $(n \geq 0, x_0 \in [a, b])$ it follows that $f(x)$ is identically zero.

Theorem A. *A necessary and sufficient condition for $C\{M_n\}$ to be quasi-analytic is that*

$$\int_1^\infty \frac{\log T(r)}{r^2}\, dr = \infty,$$

where

$$T(r) = \underset{1 \leq n}{\text{l.u.b.}} \frac{r^n}{M_n}.$$

This is Denjoy-Carleman's theorem [1]. The particular form given here is due to Ostrowski [3].

If there exists in $C\{M_n\}$ a function $f(x)$, not identically zero in $[a, b]$ and such that $f^{(n)}(x_0) = 0$ $(n \geq 0)$, there exists also a function $\varphi(x) \in C\{M_n\}$ not identically zero and such that $\varphi^{(n)}(a) = \varphi^{(n)}(b) = 0$ $(n \geq 0)$ [6]. Thus Theorem A may be stated in the following manner:

Theorem B. *A necessary and sufficient condition that there exist a function $\varphi(x) \in C\{M_n\}$, not identically zero and such that $\varphi^{(n)}(a) = \varphi^{(n)}(b) = 0$ $(n \geq 0)$, is that*

$$\int_1^\infty \frac{\log T(r)}{r^2}\, dr < \infty.$$

The present author has introduced the notion of quasi-analyticity for functions

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