A PRINCIPLE OF JESSEN AND GENERAL FUBINI THEOREMS

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Much of Jessen's fundamental work on direct product spaces, in particular the existence of measure, has in recent years been given a fairly general form. However, there is one important principle discovered by Jessen that has not been announced in the abstract form it deserves, and we propose in this note to take a step in that direction. The principle referred to is embodied in the theorem asserting that a real measurable function $f(x)$ on a direct product space, which is independent of any change of a finite number of the coördinates of x, is necessarily almost everywhere a constant. This theorem has many applications, but unfortunately the Euclidean methods of proof used by Jessen cannot be extended very far. We shall show here that the same theorem is valid even if the measure has its values in a Hausdorff ring with only two idempotents and the function f has its values in a separable metric space. We also develop the Fubini-Jessen theorems for vector-valued functions.

Notation and terminology. The letter A will be used for a class of elements α. B will always mean a non-void subset of A and σ a non-void finite subset of A. The symbols $\tilde{B}$, $\tilde{\sigma}$ will be used for the complements of B, σ in A. The symbol x/B will stand for a function defined on B. For each $\alpha \epsilon A$ there is a space S_α and a Borel field $\mathfrak{F}_\alpha$ of sets in S_α and $S_\alpha \epsilon \mathfrak{F}_\alpha$. We shall use the logical notation of Kuratowski, *Topologie* I, and define the direct product of sets $E_\alpha \epsilon S_\alpha$ by

$$\underset{\alpha \epsilon B}{P} E_\alpha \equiv \underset{x/B}{\mathcal{E}} \prod_{\alpha \epsilon B} [x(\alpha) \epsilon E_\alpha].$$

We shall sometimes write S^B in place of $\underset{\alpha \epsilon B}{P} S_\alpha$, and here and elsewhere the superscript B will be omitted in case $B = A$, so that whenever B and $\tilde{B}$ appear as superscripts in an equation we know that $\tilde{B} \neq 0$. Since x/B and $y/\tilde{B}$ determine uniquely a function $z = (x, y)$ on A by the convention $z(\alpha) = x(\alpha)$ or $y(\alpha)$ according as $\alpha \epsilon B$ or $\alpha \epsilon \tilde{B}$ and conversely z/A determines uniquely x/B and $y/\tilde{B}$, we may and shall sometimes write $S = S^B \times S^{\tilde{B}}$. An *elementary set in* S^B is by definition one of the form

$$E = \underset{\alpha \epsilon \sigma}{P} E_\alpha \times S^{\tilde{\sigma} B},$$

where $E_\alpha \epsilon \mathfrak{F}_\alpha$ and $\sigma \epsilon B$. The Borel field in S^B determined by the elementary sets in S^B will be denoted by $\mathfrak{F}^B$ and we write $\mathfrak{F}$ in place of $\mathfrak{F}^A$. For a set $M \subseteq S$ the projection of M on S^B is defined as

$$\mathrm{Proj}_B\, M \equiv \underset{x/B}{\mathcal{E}} \sum_{y/\tilde{B}} [(x, y) \epsilon M].$$

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