A NEW PROOF OF A THEOREM OF MENCHOFF

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1. Let $f(x)$ be a function of integrable square, of period 2π, and let $S_n(x)$ be the sum of order n of its Fourier series. It has been shown in a previous paper[1] that the study of the integral

$$\int_0^{2\pi} S_{n(x)}(x)\, dx \qquad (n(x) \leqq n)$$

considered by Kolmogoroff and Seliverstoff for the study of the almost-everywhere convergence of the series is equivalent to the study of the sum

$$T_n = \sum_1^n \left[\left(\int_0^{2\pi} f_p \cos px\, dx \right)^2 + \left(\int_0^{2\pi} f_p \sin px\, dx \right)^2 \right],$$

$\{f_p(x)\}$ being a sequence of characteristic functions of sets such that $f_p \geqq f_{p+1}$; it has been proved in the same paper that $T_n < C \log n$, C being an absolute constant.[2]

The purpose of this paper is to consider sums of the type

$$\sum_1^n \left(\int_0^1 f_p \varphi_p\, dx \right)^2,$$

$\{f_p(x)\}$ being a decreasing sequence of characteristic functions of sets, and $\{\varphi_p(x)\}$ any family of orthogonal normal functions in (0, 1). The study of such sums will be carried out as an application of Bessel's inequality in a two-dimensional domain.

2. Let $p_1, p_2, \cdots, p_n$ and P be functions of a certain number of variables $x_1, x_2, \cdots, x_k$ defined in a k-dimensional domain D, and let us write, for the sake of brevity, $\int Q\, d\tau$ instead of $\int_D Q(x_1, \cdots, x_k)\, dx_1 \cdots dx_k$. Then, assuming that

$$(1) \qquad \int p_i p_j\, d\tau = 0 \qquad (i \neq j),$$

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[1] See Comptes Rendus Acad. Sci. Paris, vol. 205(1937), pp. 14–16. See also Salem, *Essais sur les Séries Trigonométriques*, Paris, 1940.

[2] In this paper C means any absolute constant, not necessarily the same throughout the paper.