FUNCTIONS OF BOUNDED VARIATION IN TWO VARIABLES

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1. In a recent paper on non-absolutely convergent integrals, R. L. Jeffery gave the definition of a class V_1 of functions on the rectangle $R:(a, b; c, d)$ $(a \leqq x \leqq c, b \leqq y \leqq d)$.[1] A function $F(x, y)$ is in class V_1 on R if there exist on R a single-valued function $f(x, y)$ and a sequence of summable functions $s_n(x, y)$ tending to $f(x, y)$, such that $\int_e s_n(x, y)\, dx\, dy$ is bounded for all values of n and all measurable sets $e \subset R$, and

$$F(x, y) = \lim_{n \to \infty} \int_{(a,b;\,x,y)} s_n(x, y)\, dx\, dy \tag{A}$$

for all points (x, y) on R. To save repetition, we shall take R as the fundamental rectangle throughout this paper. All points and sets mentioned will be understood to lie on R; all functions will be taken as defined over R, and all statements regarding summability, measurability, functional class, and so forth, will be understood to apply over R.

It was shown in Theorem II of J that any function in class V_1 is in the class H of functions of bounded variation in the Hardy-Krause sense. The precise relationship of V_1 to the various definitions of bounded variation that have been proposed was, however, not determined; it is part of our present purpose to show that a slightly modified form of V_1 is equivalent to the class V of Vitali, which includes H.

A further definition was given of functions in class V_2. This is entirely similar to the above, except that the condition that $\int_e s_n(x, y)\, dx\, dy$ be bounded in n and e is not imposed. It was shown in Lemma VIII and Theorem XI of J that if $F(\omega)$ is a continuous function of intervals ω for which the strong derivative $F'_s(x, y)$ is finite everywhere, then there exists a sequence of functions $s_n(x, y)$ tending to $F'_s(x, y)$ for which $\int_{(a,b;\,x,y)} s_n(x, y)\, dx\, dy$ tends to $F(a, b; x, y)$. In other words $\phi(x, y) = F(a, b; x, y)$ is in class V_2 with $f(x, y) = F'_s(x, y)$. The question was raised whether the strong derivative could be replaced by the ordinary derivative in case the strong derivative was not known to exist. We answer this question by proving that given $F(\omega)$ a continuous additive function

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[1] R. L. Jeffery, *Functions of bounded variation and non-absolutely convergent integrals in two or more dimensions*, this Journal, vol. 5(1939), pp. 753–774. Here the point (a, b) is taken as (0, 0). This paper will be referred to as J.