A THEOREM OF BOAS

By Norman Levinson

Boas has proved the following theorem.[1]

Theorem. *If the entire function $f(z)$ satisfies*

$$\limsup_{|z|\to\infty} \frac{1}{|z|} \log |f(z)| < \log 2 \tag{1}$$

and $f(z)$ is not a polynomial, an infinite number of derivatives of $f(z)$ are univalent in the unit circle $|z| \leqq 1$.

Here we shall give a direct and quite simple proof of this theorem. Incidentally, as has been pointed out to me by Boas, this also furnishes a simple proof for a theorem of Takenaka. Takenaka's theorem[2] states that *if every derivative of an entire function $f(z)$ has a zero inside or on the unit circle and if* (1) *holds, then $f(z)$ is a constant.*

Obviously, Takenaka's theorem is an immediate consequence of the above stated theorem of Boas.

We now turn to the proof of the theorem of Boas.[3] By a trivial change of variable it will suffice to show that

$$\limsup_{|z|\to\infty} \frac{1}{|z|} \log |f(z)| < 1 \tag{2}$$

implies that an infinite number of derivatives of $f(z)$ are univalent in $|z| < \log 2$.

Let the power series for $f(z)$ be

$$\sum_0^\infty a_n z^n.$$

From the Cauchy integral formula for a_n and from (2) it follows that

$$|a_n| \leqq \frac{e^{(1-\epsilon)R}}{R^n}$$

for large R. In particular, if $R = n$,

$$a_n = O\left(\frac{e^n e^{-\epsilon n}}{n^n}\right) = o\left(\frac{1}{n!}\right).$$

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[1] R. P. Boas, *Univalent derivatives of entire functions*, this Journal, vol. 6(1940), p. 719.

[2] S. Takenaka, *On the expansion of integral transcendental functions in generalized Taylor series*, Proc. Physico-Math. Soc. Japan, vol. 14(1932), pp. 529–542.

[3] **Added February 10, 1941.** In a letter to Boas dated January 1, 1941 Pólya communicated the same proof as is given here.