PROJECTIONS IN MINKOWSKI AND BANACH SPACES

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Introduction. In his now classic *Théorie des Opérations Linéaires* Banach proposed the following problem: Given any closed linear subspace of a Lebesgue function space L_p (or of a sequence space l_p), $1 < p \neq 2$, does there always exist a complementary closed linear subspace? Or, equivalently, does there always exist a *projection* on any closed linear subspace of L_p or of l_p? The question has recently been answered, in the negative, by F. J. Murray [6].[1]

Bohnenblust [3] investigated projections of n-dimensional Minkowski spaces on $(n - 1)$-dimensional subspaces, with a view toward illuminating the question of the existence of projections in general Banach spaces. In this paper we take further steps in this direction.[2]

We first obtain, after necessary preliminaries to later general considerations (§1), in §2 the results of Murray by a briefer method, and in addition quantitative information which Murray did not obtain. In §3 we discuss orthogonal projections, and apply the results to obtain further quantitative information. Various generalizations of l_p-spaces are then introduced. In §4, we study a class of Banach spaces S, of which the elements are infinite sequences $x = \{x_i\}$, and which have the following symmetry property: If $x = \{x_i\}$ is any element of S, then $\{| x_i |\}$ is also an element of S, and $|| \{x_i\} || = || \{| x_i |\} ||$. These spaces include Banach spaces with a base $\{X_i\}$ having the corresponding symmetry property: if $x = \sum_{i=1}^{\infty} x_i X_i$ is the expansion of an element, then $\sum_{i=1}^{\infty} | x_i | \cdot X_i$ is an element, and $|| x || = || \sum_{i=1}^{\infty} | x_i | \cdot X_i ||$. In any space S, a Euclidean norm $|| x ||_2$ is introduced on a certain dense linear subset, and it is shown that if a projection exists for every closed linear subspace, then the Euclidean radii of the unit sphere of S in certain directions must be bounded both from 0 and from ∞. In particular, if for a space S these directions are "minimal" or "maximal", this is sufficient to require the space to be isomorphic to Hilbert space.

In §5, we study a type of spaces S which are generated by two-dimensional norms, in particular, spaces defined by a sequence $p_2, p_3, p_4, \cdots$ of exponents. These spaces specialize to l_p-spaces in case $p = p_2 = p_3 = \cdots$. Finally, in §6,

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The writer wishes to acknowledge his indebtedness to Professor Bohnenblust, both for suggesting the original problem of this investigation, and for stimulation and help received in our many discussions.