THE FRACTIONAL DERIVATIVE OF A LAPLACE INTEGRAL

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Introduction. The integral

$$\int_0^\infty e^{-xt}\, d\alpha(t), \tag{1}$$

where $\alpha(t)$, a function of the real variable t, is of bounded variation in $(0, R)$ for every positive R, has been exhaustively studied by D. V. Widder [9, 10, 11, 12].[1] In the present paper, we shall consider the integral

$$\int_0^\infty e^{-xt} t^\rho\, d\alpha(t), \tag{2}$$

where $\alpha(t)$ is as described above, and ρ is a positive constant, restricting ourselves to the case where x and $\alpha(t)$ are real.

In the case of the integral (1), $\alpha(t)$ is said to be normalized if

$$\alpha(0) = 0, \qquad \alpha(t) = \frac{\alpha(t+) + \alpha(t-)}{2} \qquad (0 < t < \infty). \tag{3}$$

In the present case, we may also take $\alpha(0+) = 0$. For let $\beta(0) = 0$, $\beta(t) = \alpha(0+)$ for $0 < t$, and set $\alpha^*(t) = \alpha(t) - \beta(t)$ for $0 \leqq t$. Since the integral

$$\int_0^\infty e^{-xt} t^\rho\, d\beta(t) \tag{4}$$

is obviously convergent for all real x and has the value zero, it is clear that wherever the integral (2) converges we have

$$\int_0^\infty e^{-xt} t^\rho\, d\alpha(t) = \int_0^\infty e^{-xt} t^\rho\, d\alpha^*(t). \tag{5}$$

The function $\alpha(t)$ will always be taken as satisfying the conditions (3) and the condition $\alpha(0+) = 0$; and any function satisfying these conditions will be said to be normalized.

The derivatives of the function defined by a convergent integral of the form (1) are given by the integrals ([9], p. 702)

$$(-1)^k \int_0^\infty e^{-xt} t^k\, d\alpha(t) \qquad (k = 1, 2, \cdots), \tag{6}$$

which leads one to expect that, if ρ is non-integral, (2) is either the fractional derivative of order ρ of (1) or its negative. Integrals of the form (1) con-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.