CHARACTERISTIC FUNCTIONS OF FAMILIES OF SETS

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In an interesting paper entitled *The characteristic function of a sequence of sets and some of its applications*, Fundamenta Mathematicae, vol. 31(1938), pp. 207–233 (see also Fundamenta Mathematicae, vol. 26(1935), p. 302) Szpilrajn has employed the characteristic function to develop a certain method of dealing with the algebraic structure of sequences of sets; and has established with the aid of this method a variety of specific theorems and equivalences in the domain of set-theoretical topology. He attributes to Kuratowski the first use of the characteristic function of a sequence of sets.

In the present note, I shall trace certain connections between the content of Szpilrajn's paper and the general theory of abstract Boolean algebras which I have developed in two memoirs published elsewhere: *The theory of representations for Boolean algebras*, Transactions of the American Mathematical Society, vol. 40(1936), pp. 37–111 (cited here by the letter R); and *Applications of the theory of Boolean rings to general topology*, ibid., vol. 41(1937), pp. 375–481 (cited here by the letter A). In doing so, I deem my chief purpose to be that of reconciling two independent points of view which prove, upon examination, to present a considerable similarity so far as the theory of the algebraic structure of sequences of sets is concerned.

As I shall point out below, an obvious but theoretically desirable generalization of Szpilrajn's work leads to the introduction of the characteristic function of an arbitrary transfinite sequence, or well-ordered family, of sets. It seems to me of more importance, perhaps, to observe that the rôle of order, which is essential to the definition of the characteristic function, appears to be artificial so far as the majority of applications is concerned. In principle, therefore, one is tempted to seek an order-free theory of the algebraic relations envisaged. I shall show here that such a theory is already in existence and that, through the adjunction of elementary considerations of order, it leads back to the theory of the characteristic function due to Szpilrajn.

1. **The space $\mathfrak{B}_\mathfrak{c}$.** If $\mathfrak{c}$ is any cardinal number, we shall denote by $\mathfrak{B}_\mathfrak{c}$ the Cartesian product of $\mathfrak{c}$ two-point Hausdorff spaces. It is a totally-disconnected bicompact Hausdorff space; in the particular case where $\mathfrak{c} = \aleph_0$ it is homeomorphic with the Cantor discontinuum. In the sequel we shall suppose that $\mathfrak{c}$ is an infinite cardinal.

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