ADDITIVE PRIME NUMBER THEORY IN REAL QUADRATIC FIELDS

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1. **Introduction.** Recent years have seen repeated decisive steps made toward a solution of the Goldbach problem. For almost two centuries this most difficult problem of additive number theory was intractable. Finally, in 1922 Hardy and Littlewood[1] introduced a powerful new method into analysis and proved on the basis of an unproved conjecture about the Riemann zeta-function that every sufficiently large odd number can be represented as the sum of three odd primes and that "almost" every even number is the sum of two primes. In 1930 Schnirelmann,[2] employing an ingenious modification of the Viggo Brun method, proved directly that every even integer can be represented as a sum of not more than 800,000 primes. The number 800,000 was lowered to 2,208 by Romanoff[3] in 1935, to 71 by Heilbronn, Landau and Scherk[4] in 1936, and to 67 by Ricci[5] in 1937. In the same year Vinogradow[6] combined the Hardy-Littlewood method with a new method of his own and gave the first complete proof that every sufficiently large odd number is the sum of three primes. Later, Estermann,[7] extending the ideas initiated by Vinogradow, proved that "almost" every even integer is the sum of two primes.

Less attention has been paid to the problem of representing numbers in an algebraic field as the sum of primes. Indeed, the only contributions in this

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[1] *Some problems of "Partitio Numerorum", III: On the expression of a number as a sum of primes*, Acta Mathematica, vol. 44(1922), pp. 1–70; *Some problems of "Partitio Numerorum", V: A further contribution to the study of Goldbach's problem*, Proceedings of the London Mathematical Society, (2), vol. 22(1923), pp. 46–56; see also E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, Leipzig, 1927, pp. 183–234.

[2] *On additive properties of numbers* (Russian with a French résumé), Ann. Inst. Polytechn. Novočerkassk, vol. 14(1930), pp. 3–28.

[3] *On the Goldbach problem* (Russian), Mitt. Forch.-Inst. Math. u. Mech. Univ. Tomsk, vol. 1(1935), pp. 34–38.

[4] *Alle grossen ganzen Zahlen lassen sich als Summe von höchstens* 71 *Primzahlen darstellen*, Casopis pro Pěstováni Matematiky a Fysiky, vol. 65(1936), pp. 117–140.

[5] *Su la congettura di Goldbach e la constante di Schnirelmann*, Annali della Scuola Normale Superiore di Pisa, (2) vol. 6(1937), pp. 91–116.

[6] *Representation of an odd number as a sum of three primes*, Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 15(1937), pp. 169–172.

[7] *On Goldbach's problem: Proof that almost all even positive integers are sums of two primes*, Proceedings of the London Mathematical Society, (2), vol. 44(1938), pp. 307–314; Independent proofs have also been given by J. G. van der Corput, *Sur l'hypothèse de Goldbach pour presque tous les nombres pairs*, Acta Arithmetica, vol. 2(1937), pp. 266–290, and by Tchudakoff, *On the density of the set of even numbers which are not representable as a sum of two primes*, (Russian), Bull. Acad. Sci. de l'URSS, Ser. Math. No. 1, vol. 40(1938), pp. 25–39.