REMARK ON A RECENT PAPER OF O. ORE

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O. Ore, in a recent paper,[1] studies, among other things, the groups he describes as "groups with conformal chains". After examining the principal properties of these groups, he concludes his paper with the following theorem:

Let G be a group such that every subgroup and every quotient group of G have subgroups of every possible order. Then G is a group with conformal chains, and conversely.

Ore notes moreover that it would be of interest to know whether the condition on the quotient groups is necessary. The following theorem proves that the condition on the quotient groups is not necessary, because it is a consequence of the condition on the subgroups.

THEOREM. *Let G be a group such that every (proper or improper) subgroup of G has subgroups of every possible order. Let N be a normal subgroup of G. Then every (proper or improper) subgroup of $\bar{G} = G/N$ has subgroups of every possible order.*

Proof. Let g be the order of G. Each group whose order is a prime, or a product of two equal or different primes, has subgroups of every possible order; hence the theorem is true for these groups. We shall prove the theorem by induction with respect to the number of prime factors of g. Suppose the theorem true for each group whose order divides g. The groups G and $\bar{G}$ are multiply isomorphic. Let $\bar{M}$ be a proper subgroup of $\bar{G}$; and let M be the (proper) subgroup of G whose operations correspond to operations of $\bar{M}$ in multiple isomorphism. We have $\bar{M} = M/N$. Since M is in G, M and every subgroup of M have subgroups of every possible order. The conditions of the theorem are then verified for M. Since the order of M divides g, the theorem is true for M, and consequently $\bar{M}$ (and with $\bar{M}$ every proper subgroup of G) has subgroups of every possible order.

It remains to demonstrate that $\bar{G}$ also has subgroups of every possible order. Let n be the order of N. The order of $\bar{G}$ is then $\bar{g} = g/n$. Let $\bar{d}$ be a divisor of $\bar{g}$. We shall prove that $\bar{G}$ has at least a subgroup of order $\bar{d}$. Let p be a prime dividing $\bar{g}/\bar{d}$. Then $\bar{d}$ divides $\bar{g}/p$. If $\bar{G}$ has a subgroup $\bar{S}$ whose order is $\bar{g}/p$, $\bar{S}$ has subgroups of every possible order and hence of order $\bar{d}$; and $\bar{S}$ being in $\bar{G}$, also $\bar{G}$ has subgroups of order $\bar{d}$. It will then be sufficient to demonstrate that if p is a prime dividing $\bar{g}$, $\bar{G}$ has at least a subgroup of order $\bar{g}/p$.

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[1] Oystein Ore, *Contributions to the theory of groups of finite order*, this Journal, vol. 5 (1939), pp. 431–460.