CONVEXITY IN A LINEAR SPACE WITH AN INNER PRODUCT

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1. **Introduction.** In Euclidean space, a point set $\mathfrak{M}$ is said to be *linearly connected* (or convex) if, when $x \equiv (x_1, x_2, \cdots, x_n)$ and $y \equiv (y_1, y_2, \cdots, y_n)$ are any two points of the set, all points on the segment joining x and y also belong to the set. Analytically:

(1) *If x and y are in $\mathfrak{M}$, then $kx + (1 - k)y$ is in $\mathfrak{M}$ for $0 \leqq k \leqq 1$.*

A line (in two-dimensional Euclidean space) or a hyperplane (in n-dimensional Euclidean space) is said to be a *supporting* line or hyperplane of a set $\mathfrak{M}$ if it passes through at least one point of $\mathfrak{M}$ and does not separate the points of $\mathfrak{M}$. A set $\mathfrak{M}$ such that through each of its boundary points there can be passed a supporting hyperplane is said to be *completely supported at its boundary points.*

In Euclidean space, the following theorems are well known.[1]

THEOREM A. *If a point set is linearly connected, it is completely supported at its boundary points.*

THEOREM B. *If a point set is closed, possesses inner points, and is completely supported at its boundary points, then it is linearly connected.*[2]

The purpose of this paper is to examine how far these theorems persist in a more general space. Obviously their validity will depend on the space and the definitions assumed.

We center our attention on a space $\mathfrak{S}$ which is *linear* and in which an inner product is defined. That is, for any two elements x and y of our space there is assumed to exist a unique real-valued number denoted by $((x, y))$ and called the *inner product* of x and y. This inner product is assumed to satisfy the following:

$$((cx, y)) = c((x, y)) \qquad \text{for every real number } c,$$

$$((x + y, z)) = ((x, z)) + ((y, z)),$$

$$((x, y)) = ((y, x)),$$

$$((x, x)) \geqq 0,$$

with the equality holding if and only if x is the zero element of our space, and

$$| ((x, y)) | \leqq ((x, x))^{\frac{1}{2}} \cdot ((y, y))^{\frac{1}{2}}.$$

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[1] See, for example, American Mathematical Monthly, vol. 45(1938), p. 202.

[2] For a symmetrical theorem, but one which is weaker than Theorems A and B, we have the following: *If a set $\mathfrak{M}$ is closed and possesses inner points, then linear connectedness of the set $\mathfrak{M}$ implies complete support at its boundary, and conversely.*