ISOMETRIC EMBEDDING OF FLAT MANIFOLDS IN EUCLIDIAN SPACE

By C. Tompkins

1. If $x_1, \cdots, x_{2n}$ represent a set of normal orthogonal coördinates in a Euclidian space of $2n$ dimensions, and if u_a $(a = 1, 2, \cdots, n)$ is a set of parameters which may be reduced modulo 2π, the equations

$$x_{2a-1} = \cos u_a, \qquad x_{2a} = \sin u_a \tag{1}$$

represent an n-dimensional torus (the product of n circles) embedded in the Euclidian space. Furthermore, the metric induced on this torus is flat; with the given choice of parameters it is clear that $g_{ab} = \delta_{ab}$, where g_{ab} represents the fundamental metric tensor induced on the torus and δ_{ab} is zero if a differs from b and one if a and b are equal. This shows that there exists a flat n-dimensional manifold which may be sometrically embedded in $2n$-dimensional Euclidian space.

The object of this note is to show that it is impossible to embed a closed n-dimensional manifold in Euclidian space of less than $2n$ dimensions so that (1) the manifold admits a representation by Frenet equations with coefficients satisfying Gauss equations[1] and (2) the metric induced on the manifold is flat. This result is already known for the case where n is two.[2]

For purposes of convenience, indices chosen from the first part of the alphabet will have the range from 1 to n, and those from the last part have the range 1 to $(n - 1)$. Latin indices, even when repeated, are not summed. Greek indices are summed.

2. **Algebraic lemmas.** Let A be a matrix of n rows and n columns whose elements a_{ab} are vectors in a real space of a fixed number of dimensions. A will be said to satisfy condition I if and only if

$$a_{ac}\,a_{bd} = a_{ad}\,a_{bc}, \tag{2}$$

where the multiplication means the scalar product. A will be said to satisfy condition II if and only if the vanishing of any linear combination of any one diagonal element and any set of non-diagonal elements implies that the coefficient of the diagonal element is zero; i.e., each diagonal element is independent of the non-diagonal ones.

Lemma 1. *If the matrix A satisfies both conditions* I *and* II *simultaneously, the elements a_{ab} lie in a space of at least n dimensions.*

Received July 15, 1938. The author is a National Research Fellow.

[1] See H. Weyl, Mathematische Zeitschrift, vol. 12(1922), pp. 154–160; C. Tompkins, Bulletin of the American Mathematical Society, vol. 41(1935), pp. 931–936.

[2] L. P. Eisenhart, *Differential Geometry*, New York, 1909, p. 156.