THE WARING PROBLEM WITH SUMMANDS x^m, $m \geqq n$

By Mary Haberzetle

In the original Waring problem the summands considered are positive integral n-th powers. By using different types of summands one obtains various modifications of the problem. In this paper those of the form x^m, $m \geqq n$, are used. If a positive integer is a sum of u such summands with $x \geqq 0$, it is said to be a sum of u "values". If $n \geqq 9$ and $r \leqq 2^n - k - 3$, $I = 2^n + k - 1$ values are necessary and sufficient to represent every positive integer; r is defined by $3^n = 2^n q + r$, $q = [(\frac{3}{2})^n]$, and $k = [.584963n]$. The notation $[x]$ denotes the largest integer $\leqq x$. For $9 \leqq n \leqq 400$ it is shown that the inequality $r \leqq 2^n - k - 3$ is satisfied.

This problem was suggested by L. E. Dickson.

1. **Representation of integers $\leqq 2^n q$.** Write $3^n = 2^n q + r$, $1 \leqq r < 2^n$, $q = [(\frac{3}{2})^n]$.

Lemma A. *Every positive integer $< 2^x - 1$ is a sum of at most $x - 1$ terms $1, 2, \cdots, 2^{x-1}$.*

The lemma may be verified for $x = 1$, $x = 2$. Assume, therefore, that it is true for integers $< 2^x - 1$ and prove by induction. The integer $2^x - 1$ is equal to $1 + 2 + \cdots + 2^{x-1}$ and hence requires x terms. The integers $2^x + 1$, $2^x + 2, \cdots, 2^{x+1} - 2$ are each the sum of 2^x and an integer $< 2^x - 1$. Hence all $< 2^{x+1} - 1$ are sums of at most x terms $1, 2, \cdots, 2^x$.

Let k be the maximum positive integer such that $2^{n+k} \leqq 2^n q$. Then k is the largest integer for which $2^k \leqq q = [(\frac{3}{2})^n]$, that is,

$$k = [(n \log \tfrac{3}{2})/\log 2] = [.584963n].$$

Lemma 1. *All positive integers $\leqq 2^n q$ are sums of $I = 2^n + k - 1$ values. The integer $2^{n+k} - 1$ requires exactly I values.*

Let $B = 2^n x + y$, $0 \leqq y \leqq 2^n - 1$, $0 \leqq x < q$. Then $x < 2^{k+1} - 1$ and by Lemma A is a sum of at most k terms $1, 2^2, \cdots, 2^k$. Hence B is a sum of $2^n + k - 1$ values. Since $q \leqq 2^{k+1} - 1$, $2^n q$ is a sum of $k + 1$ values. The integer $2^{n+k} - 1$ is equal to $2^n(1 + 2 + \cdots + 2^{k-1}) + 2^n - 1$ and hence is a sum of $2^n + k - 1$ values and not fewer.

2. **Results from the asymptotic theory.** The following theorem, whose proof is due to Vinogradow[1] and L. E. Dickson,[2] is stated here with an improved

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[1] Vinogradow, *On Waring's problem*, Annals of Mathematics, vol. 36(1935), pp. 395–405.

[2] L. E. Dickson, *Proof of the ideal Waring theorem for exponents* 7–180, American Journal of Mathematics, vol. 58(1936), p. 525.