THE MEAN ERGODIC THEOREM

BY GARRETT BIRKHOFF

1. Introduction. Von Neumann's Mean Ergodic Theorem will be given an elementary proof below in the following generalized form:

THEOREM. *Let X be any uniformly convex Banach space, and T any linear operator on X such that $|xT| \leqq |x|$ for all x. Then the "n-th means"*

$$g_n = \frac{1}{n}(f + fT + \cdots + fT^{n-1})$$

of the transforms fT^k of any $f \epsilon X$, under powers of T, converge to a limit.

Uniform convexity is understood in the sense of J. A. Clarkson to mean that $|x| \leqq |y| \leqq 1$ and $|x - y| \geqq \epsilon$ imply $|\frac{1}{2}(x + y)| \leqq |y| - u(\epsilon)$, where $u(\epsilon) > 0$ if $\epsilon > 0$. Equivalently, $|x| \leqq |y| \leqq 1$ and $|y| - |\frac{1}{2}(x + y)| \leqq \epsilon$ imply $|x - y| \leqq w(\epsilon)$, where $w(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$. Clarkson gives[1] an elementary proof that the spaces L_p $(p > 1)$ are uniformly convex.

The author has had access to the following unpublished papers: F. Riesz, *Some mean ergodic theorems*; N. Wiener, *The ergodic theorem* (published in this issue of this Journal; S. Kakutani, *Weak convergence in uniformly convex spaces*; K. Yosida, *Mean ergodic theorem in Banach spaces.*

The author's proof is most closely related to Wiener's, which alone does not depend on the non-elementary theorem that the unit sphere in Hilbert space is weakly compact. It is more general and somewhat shorter than Wiener's, which applies only to isometric transformations of L_2, or Riesz', which is only sketched for L_p $(p \neq 2)$. It is much shorter than the proofs of Kakutani and Yosida, although less general.

2. Verbal argument. The key point is that unless g_nT^{kn} is very near g_n, $|\frac{1}{2}(g_n + g_nT^{kn})|$ must by uniform convexity be appreciably less than $|g_n|$. Hence the norm of the mean of the $\frac{1}{2}(g_n + g_nT^{kn})T^{hn}$, for $h = 0, \cdots, k - 1$, will also be appreciably less than $|g_n|$. But this mean is

$$\left|\frac{1}{2k}(g_n + g_nT + \cdots + g_nT^{2kn-n})\right| = |g_{2kn}|.$$

It follows that if $|g_n|$ is chosen sufficiently near its infimum M, then every g_nT^{kn} will be arbitrarily near g_n. Hence so will every g_{mn}, being the mean of

Received December 29, 1938.

[1] J. A. Clarkson, *Uniformly convex spaces*, Trans. Am. Math. Soc., vol. 40(1936), pp. 396–414. In Hilbert space, $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$, whence if $|x| \leqq |y|$, $|\frac{1}{2}(x + y)|^2 \leqq |y|^2 - |\frac{1}{2}(x - y)|^2$, and $|\frac{1}{2}(x + y)| \leqq |y| - \frac{1}{4}|x - y|^2$ if $|y| \leqq 1$.