ONE-PARAMETER FAMILIES OF TRANSFORMATIONS

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One-parameter families of transformations taking measurable sets into measurable sets arise in many parts of mathematics. The purpose of this paper is to present a detailed study of the regularity properties of such families. Before we begin this study, some introductory remarks on measures in product spaces will be made. These remarks have some independent interest, so they will be phrased more generally than necessary for the actual applications to be made in the present paper.

Introduction

Let T (Ω) be any abstract space, whose points will be denoted by t (ω). Let $\mathcal{F}_t$ $(\mathcal{F}_\omega)$ be any Borel field[1] of t-sets (ω-sets) including T (Ω) itself. The space of points (t, ω), the direct product of the two spaces, will be denoted by $T \times \Omega$. Let E be a set of $\mathcal{F}_t$ and let Λ be a set of $\mathcal{F}_\omega$. Then the condition $t \in E$, $\omega \in \Lambda$ determines a (t, ω)-set $E \times \Lambda$, and we shall denote by $\mathcal{F}_t \times \mathcal{F}_\omega$ the Borel field of (t, ω)-sets determined by all such sets $E \times \Lambda$.[2] If a Borel field of point sets is the Borel field determined by some denumerable subcollection, it will be called *strictly separable.* If $\mathcal{F}_t$ and $\mathcal{F}_\omega$ are strictly separable, $\mathcal{F}_t \times \mathcal{F}_\omega$ is also strictly separable. Moreover, if $\tilde{\Lambda}$ is any set of $\mathcal{F}_t \times \mathcal{F}_\omega$, there is a strictly separable subfield $\mathcal{F}'_t$ $(\mathcal{F}'_\omega)$ of $\mathcal{F}_t$ $(\mathcal{F}_\omega)$ such that $\tilde{\Lambda}$ is a set of the field $\mathcal{F}'_t \times \mathcal{F}'_\omega$.[3]

Let $\mu(\Lambda)$ be a non-negative completely additive set function defined on the field $\mathcal{F}_\omega$.[4] An ω-set Λ_1 will be called measurable if it differs from a set Λ_0 of $\mathcal{F}_\omega$ by a subset of a set of $\mathcal{F}_\omega$ of measure 0, and we define $\mu(\Lambda_1)$ as $\mu(\Lambda_0)$. Let $\mathcal{F}^*$ be the following space: a point of $\mathcal{F}^*$ is a class of measurable ω-sets, any two of which differ at most by a set of measure 0. We metrize $\mathcal{F}^*$ as follows: if Λ_1^*, Λ_2^* are points of $\mathcal{F}^*$, and if Λ_i is a set in the class Λ_i^*, the distance between Λ_1^*, Λ_2^* is defined as arc tan $\mu(\Lambda_1 + \Lambda_2 - \Lambda_1 \cdot \Lambda_2)$,[5] or $\frac{1}{2}\pi$ if $\mu(\Lambda_1 + \Lambda_2 - \Lambda_1 \cdot \Lambda_2)$

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[1] A field of sets is a collection of sets including $E_1 + E_2$, $E_1 - E_1 \cdot E_2$ if it includes E_1, E_2. A Borel field of sets is a field of sets including $\sum_1^\infty E_n$ if it includes $E_1, E_2, \cdots$.

[2] The Borel field of sets determined by a given collection of sets is the smallest Borel field of sets containing the given collection.

[3] The collection of all sets $\tilde{\Lambda}$ having this property is readily seen to be a Borel field. This Borel field certainly includes every set $E \times \Lambda$ as defined above, so the field is precisely $\mathcal{F}_t \times \mathcal{F}_\omega$.

[4] If $\mu(\Lambda)$ is not always finite-valued, we assume that Ω can be expressed as a denumerably infinite sum of sets in $\mathcal{F}_\omega$ on each of which $\mu(\Lambda)$ is finite-valued.

[5] If $\mu(\Omega) < +\infty$, we can define the distance between Λ_1^*, Λ_2^* as $\mu(\Lambda_1 + \Lambda_2 - \Lambda_1 \cdot \Lambda_2)$ and obtain the same $\mathcal{F}^*$-topology, but this definition is not possible in the general case, if the distance function is to be finite-valued.