FACTORIZATION IN PRINCIPAL IDEAL RINGS

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Factorization theorems in a ring $K[x]$ of polynomials over a field K have been obtained by several writers [1, 3].[1] We consider this problem in the present paper for associative rings of a more general type, namely, rings (with unit element) in which every left ideal is a principal left ideal. We prove that two similar elements in such a ring which are factored in some way into a product of elements in $K[x]$ possess further factorizations which are essentially alike. The relation of two-sided ideals to left ideals is also considered for special rings. The latter leads to a generalization of a theorem due to N. Jacobson [2].

1. **Examples of rings which are principal left ideal rings.** Among such are:

(a) $K[x]$, a ring of polynomials $\alpha = \sum_{i=0}^{n} k_i x^i$ $(n = 0, 1, 2, \cdots, k_n \neq 0)$ in one indeterminate x over a field K (non-commutative in general). The degree of α equals n. One assumes [3] that there is an associative and distributive (over addition) multiplication defined such that

$$\deg (\alpha \cdot \beta) = \deg \alpha + \deg \beta \tag{1}$$

for each pair α, β of elements of $K[x]$. Condition (1) implies

$$x\alpha = \bar{\alpha}x + \alpha'.$$

Cf. [2].

(b) R_n, the square matrix ring of degree n with elements from R, where R is itself a principal left ideal ring with unit element.

To show this, let I be a left ideal in R_n and consider the set of all vectors $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_k = a_{ik}$ $(k = 1, \cdots, n)$, the matrix (a_{ik}) being in I. This set forms an R-left module M when addition is defined by

$$(\alpha_1, \alpha_2, \cdots, \alpha_n) \pm (\beta_1, \beta_2, \cdots, \beta_n) = (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \cdots, \alpha_n \pm \beta_n)$$

and

$$\rho(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\rho\alpha_1, \rho\alpha_2, \cdots, \rho\alpha_n), \qquad \rho \in R.$$

The first components of all the vectors in M thus form a left ideal in R generated by some element, say α, which must therefore be the first component of some vector A_1 in M. If $\alpha = 0$, then choose A_1 to be the null vector $(0, 0, \cdots, 0)$. Every vector of M is congruent (mod A_1) to a vector in which the first com-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.