CROSS-SECTIONS OF CURVES IN 3-SPACE

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1. **Introduction.** We consider in this paper "regular families of curves", that is, families F of non-intersecting curves such that if two curves are sufficiently close at a point, then they remain close for a "finite time"; see [3] (i.e., the third reference below), Theorem 7A. It was shown in [3] that a cross-section may always be found through an arbitrary point of the family. If the family fills a region of Euclidean 3-space E, it is natural to suspect that *any cross-section contains a cross-section which is a 2-cell*; our object here is to prove this fact. It follows (see [3], §20) that *locally, a family of curves in E is equivalent to a family of straight lines.* The proof is arranged so that a minimum of preliminary material is assumed. We use rather fully the methods in [1], [2], and the first half of [3], and a method of proof in [4].

2. **Two types of homology.** We relate the two types of homology used in [1] and [2], and give some simple properties of the second type. For a curve (= simple closed curve) J in a closed set, $J \sim 0$ was defined in [2]. For J in a general set, say $J \sim 0$ if it is ~ 0 in some bounded closed subset. Call a chain of a subdivision from some fixed sequence of simplicial subdivisions (as in [1]) a *polygonal* chain. Say two curves are equivalent in a set G if, using fixed parametrizations $f_0(\theta)$ and $f_1(\theta)$ for them, one can be deformed into the other in G, i.e., $f_t(\theta)$ $(0 \leqq t \leqq 1)$ exists and is continuous.

Lemma 1. *If J and J' are equivalent in G, then $J \sim 0$ in G if and only if $J' \sim 0$ in G.*

This is easily seen, using [2], Lemma I, if we subdivide the θ-circle and the t-segment, and consider triangles of the forms

$$f_{t_i}(\theta_j)f_{t_{i+1}}(\theta_j)f_{t_i}(\theta_{j+1}), \qquad f_{t_{i+1}}(\theta_j)f_{t_i}(\theta_{j+1})f_{t_{i+1}}(\theta_{j+1}).$$

Lemma 2. *If $J \sim 0$ in A, and A is deformed into A', leaving J fixed, then $J \sim 0$ in A'.*

For if $L \to K$, K a 1-cycle in J, we need merely consider the deformed L.

Lemma 3. *Let J be equivalent to a polygonal J' in an open set G. Then $J \sim 0$ in G if and only if J' bounds a polygonal chain in G.*

By Lemma 1, $J \sim 0$ if and only if $J' \sim 0$. Suppose J' bounds a polygonal chain. Then, using a fine enough subdivision and [2], Lemma I, we see that $J' \sim 0$. Suppose $J' \sim 0$; say $J' \sim 0$ in the closed subset A of G. Set $\epsilon =$

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