# SOME GAP THEOREMS FOR POWER SERIES

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1. **Introduction.** In this note we are concerned with the behavior of certain power series,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

on the circle of convergence (which we suppose to be the unit circle). We consider, not the convergence of the series (1.1) for $|z| = 1$, but the convergence of a suitably chosen subsequence of its partial sums. It is to be expected that such a sequence may converge under hypotheses on $f(z)$ lighter than those which ensure the convergence of the series itself; but the situation is complicated by the fact that the sequence, unlike the sequence of all the partial sums, may converge without converging to the "right" value. By way of illustration, we consider three examples. We write $s_n = \sum_{k=0}^{n} a_k$.

(i) $a_n = (-1)^n$, $f(z) = 1/(1+z)$. Then $s_{2n-1} \to 0$, but $f(z) \to \frac{1}{2}$ as $z \to 1$.

(ii) $a_0 = 1$, $a_{2m} = -a_{2m-1} = 4m + 1$ $(m = 1, 2, \cdots)$; $f(z) = 2(1+z)^{-2} - (1-z)^{-1}$. Here $s_{2n} \to 1$, but $|f(z)| \to \infty$.

(iii) $a_{3n} = a_{3n+2} = 1$, $a_{3n+1} = -2$ $(n = 0, 1, 2, \cdots)$; $f(z) = (1-z)^2/(1-z^3)$. Then $s_{3n-1} \to 0$ and $f(z) \to 0$.

Apparently the only results in the literature concerning the convergence of subsequences of partial sums are those of A. Ostrowski. Ostrowski's theorem[1] states that if there are infinitely many sufficiently long gaps in the sequence $\{a_n\}$ (if, in fact, $a_n = 0$ for $n_k < n < n_k(1+\epsilon)$, where $\epsilon > 0$ and $n_k \to \infty$), then regularity of $f(z)$ on a closed arc of $|z| = 1$ implies the uniform convergence of the sequence $s_{n_k}(z) = \sum_{n=0}^{n_k} a_n z^n$ to $f(z)$ in a domain containing that arc—in particular, then, on the arc. Our theorems resemble the result of Ostrowski in assuming the existence of gaps in the sequence $\{a_n\}$; they assume less than the regularity of $f(z)$ at points of the circle of convergence, but require restrictions on the rate of growth of the $a_n$. Naturally, we obtain convergence of $\{s_{n_k}(z)\}$ only for points of $|z| = 1$, not for exterior points. In one theorem (Theorem 2) we require $f(z)$ to be of bounded variation on an arc of $|z| = 1$; we obtain convergence of $\{s_{n_k}(z)\}$ on the arc under the assumption of smaller gaps than those necessary for Ostrowski's theorem. This result extends a theorem of P. Fatou,[2] which states that $a_n = o(1)$ implies convergence of (1.1) at every

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[1] Dienes [2], p. 358.

[2] Dienes [2], p. 467.