INTEGRAL FUNCTIONS BOUNDED ON SEQUENCES OF POINTS

By Norman Levinson

1. As an extension of results of V. Ganapathy Iyer we shall prove the following

Theorem. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two increasing sequences such that*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D_\lambda, \qquad \lim_{n\to\infty} \frac{n}{\mu_n} = D_\mu.$$

Let the indices of condensation[1] *of these sequences be zero. Let $f(z)$ be an integral function such that*

$$f(\pm\lambda_n) = O(1), \qquad f(\pm i\mu_n) = O(1), \tag{1.0}$$

and

$$\varlimsup_{|z|\to\infty} \frac{\log |f(z)|}{|z|} = k < \pi(D_\lambda^2 + D_\mu^2)^{\frac{1}{2}}. \tag{1.1}$$

Then $f(z)$ is a constant.

That the above theorem is a best possible result follows from trivial considerations (for example, $f(z) = \sin \pi D_\lambda z \sinh \pi D_\mu z$). Iyer[2] has proved that the above result holds under more stringent conditions; namely, with (1.1) replaced by

$$k < \pi \min (D_\lambda, D_\mu), \tag{1.2}$$

or else with (1.0) replaced by

$$\lim_{n\to\infty} \frac{\log |f(\pm\lambda_n)|}{\lambda_n} = \lim_{n\to\infty} \frac{\log |f(\pm i\mu_n)|}{\mu_n} = -\infty.$$

The method of proof used here is an extension of a method used in proving certain simpler theorems.[3]

2. The proof of our theorem will be given in two parts. Part 1 is the essential part.

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[1] The term index of condensation is defined in *Séries de Dirichlet*, Vladimir Bernstein, Paris, 1933, p. 25. $\varliminf_{n\to\infty} (\lambda_{n+1} - \lambda_n) > 0$ is sufficient for zero index of condensation.

[2] V. Ganapathy Iyer, *On the order and type of integral functions bounded at a sequence of points*, Annals of Mathematics, vol. 38 (1937), p. 311.

[3] N. Levinson, *On certain theorems of Pólya and Bernstein*, Bulletin of the American Mathematical Society, vol. 42 (1936), p. 702. The method is an extension of that used in proving Theorem 2, p. 703.