LINEAR FUNCTIONALS SATISFYING PRESCRIBED CONDITIONS

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1. **Introduction.** A function (or transformation) $q \equiv q(x)$ with domain and range in linear spaces is called *linear* if

$$q(ax + by) = aq(x) + bq(y) \qquad (a, b \in R;\ x, y \in E), \tag{1.01}$$

where E is the domain of q and R is the set of real numbers. If the range of $q(x)$ is in R, then $q(x)$ is called a *functional.* Using notation of Banach[1] we call a functional $p(x)$ a *p-function* if

$$p(tx) = tp(x) \qquad (t \geqq 0;\ x \in E), \tag{1.02}$$

$$p(x + y) \leqq p(x) + p(y) \qquad (x, y \in E). \tag{1.03}$$

We denote the class of linear functionals $f \equiv f(x)$ by F and the class of p-functions by P.

A theorem of Banach (loc. cit., p. 29) of which we make repeated use is

THEOREM 1.1. *If $p \in P$, then there exists $f \in F$ with*

$$f(x) \leqq p(x) \qquad (x \in E). \tag{1.11}$$

Since each linear functional f is also a p-function, i.e., $F \subset P$, the following theorem, of which we shall make explicit and implicit use, is trivial.

THEOREM 1.2. *If $f \in F$, then there exists $p \in P$ with $f(x) \leqq p(x)$ for all $x \in E$.*

Let $p_0 \in P$ and a set Ψ of pairs $\{x, y\}$ of elements $x, y \in E$ be prescribed. One problem in which we shall be interested is that of determining whether there exist linear functionals $f \in F$ possessing the properties

$$f(x) \leqq p_0(x) \qquad (x \in E), \tag{1.21}$$

$$f(y) = f(x) \qquad (\{x, y\} \in \Psi). \tag{1.22}$$

We assume Ψ has the property that if $\{x, y\} \in \Psi$ then $\{y, x\} \in \Psi$, and that $\{x, x\} \in \Psi$ for each $x \in E$; this assumption is convenient and entails no loss of generality.

We shall say that a p-function $p \equiv p(x)$ *enforces* a specified property (or set of properties) if every $f \in F$, with $f(x) \leqq p(x)$ for all $x \in E$, must possess the specified property (or set of properties).

For example, a slight amplification of work of Banach (loc. cit., p. 33) shows

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[1] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 28.