ON SOME GENERALIZATIONS OF A THEOREM OF A. MARKOFF

BY EINAR HILLE, G. SZEGÖ, AND J. D. TAMARKIN

I. Introduction

1. The theorems of A. Markoff and of S. Bernstein concerning the derivative of a rational or of a trigonometric polynomial state that if $||f||$ denotes the maximum of the absolute value of a rational polynomial $f(x)$ over a finite interval (a, b), or of a trigonometric polynomial over its interval of periodicity, then for the derivative $f'(x)$ we have

$$(1.1) \qquad ||f'|| \leqq An^2 ||f|| \quad \text{or} \quad ||f'|| \leqq An ||f||,$$

respectively, where n is the degree of $f(x)$ and A is a constant which does not depend on n or on f, but only on $(b - a)$. In fact $A = 1$ in the case of rational polynomials considered on $(-1, +1)$ and also in the case of trigonometric polynomials of period 2π. These results can be stated in "abstract" form if we consider $f(x)$ as an element of the space C of continuous functions and interpret $||f||$ as the "norm" of this element. A natural question arises then whether estimates similar to (1.1) hold if $f(x)$ is considered as an element of other function spaces with different definition of the norm. The purpose of the present note is to answer this question for rational polynomials in the case of the space L_p, $p \geqq 1$, where the norm is defined by

$$||f|| = ||f||_p = \left\{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx\right\}^{1/p}.$$

2. The corresponding problem for trigonometric polynomials was solved in a much more general case by Zygmund[1] by using an important interpolation formula of M. Riesz.[2] According to this formula we have, for an arbitrary trigonometric polynomial of degree n and of period 2π,

$$(1.2) \qquad |f'(x)| \leqq \sum_{\nu=1}^{2n} \rho_\nu^{(n)} |f(x + \theta_\nu^{(n)})|,$$

where $\rho_\nu^{(n)}$, $\theta_\nu^{(n)}$ are certain numbers which do not depend on $f(x)$ and which satisfy

$$(1.3) \qquad \rho_\nu^{(n)} > 0, \qquad \sum_{\nu=1}^{2n} \rho_\nu^{(n)} = n,$$

$$(1.4) \qquad 0 < \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_{2n}^{(n)} < 2\pi.$$

Received October 2, 1937.

[1] *A remark on conjugate functions*, Proceedings of the London Math. Soc., (2), vol. 34 (1932), pp. 392–400, esp. pp. 394–396.

[2] *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Math. Verein., vol. 22 (1914), pp. 354–368, esp. p. 356, (9), (10).