RESIDUATION IN STRUCTURES OVER WHICH A MULTIPLICATION IS DEFINED

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I. Introduction

1. Consider a set of elements A, B, C, $\cdots$ forming an abstract structure[1] Σ over which there is defined a commutative and associative multiplication XY. The multiplication operation is connected with the structure by assuming that it is distributive with respect to union:

$$A(B, C) = (AB, AC) \qquad \text{for } A, B, C \text{ in } \Sigma. \tag{3.3}$$

This condition is satisfied in the important instance of the ideals of a commutative ring.

If we assume that any subclass of elements of Σ has a union and correspondingly strengthen assumption (3.3), the existence of a *residual* (German, *ideal-quotient*[2]) $A:B$ follows for each pair of elements A, B of Σ with the defining properties

$$A \supset (A:B)B; \qquad \text{if } A \supset XB, \text{ then } A:B \supset X.$$

It is easy to show that the residual thus defined has the formal properties of the residual in polynomial ideal theory (Macaulay, [3]). But in the special instance of ordinary arithmetic (when Σ is interpreted as the ring of rational integers) the residual has a number of additional interesting properties which do not hold in general; for example,

$$(A:B, B:A) = I,^{3} \qquad (A, B):M = (A:M, B:M),$$

$$M:[A, B] = (M:A, M:B), \qquad M:AB = \{(M:A)(M:B)\}:M,$$

$$A:(B:A) = A(A, B):B, \qquad [A, B]:(A, B) = (A:B)(B:A).$$

The problem arises then of determining the conditions under which these and other properties of residuation in ordinary arithmetic will hold in the abstract structure. It is not difficult to show that it suffices to assume[4]

Postulate E. *If A divides B, there exists a unique element Q such that $AQ = B$.*

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[1] Other terms are "dual group", "Verband", "lattice". For a definition, see §2 of this paper, or O. Ore, reference [1] at the close of the paper.

[2] The concept appears to be due to Dedekind [4]. See van der Waerden [2] or Macaulay [3].

[3] I here is the unit element with respect to multiplication. See §§2 and 3.

[4] Postulate E is satisfied in every principal ideal ring.