NON-n-ALTERNATING TRANSFORMATIONS

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Let A and B be compact metric spaces and $T(A) = B$ a single-valued continuous transformation. We shall say that T is *non-n-alternating* provided that, for any point x of B for which there exists a cutting K of $A - T^{-1}(x)$ consisting of at most n points, there is no point y of B such that $T^{-1}(y)$ intersects both sets of the separation $A - (T^{-1}(x) + K) = A_1 + A_2$. If K is the null set, this is the definition of a non-alternating transformation.[1] Consequently, this type of transformation is non-alternating; in fact, we have the following characterization:

Theorem I. *A necessary and sufficient condition that a single-valued continuous transformation $T(A) = B$ be non-n-alternating is that T be non-alternating on the complement of every subset of A consisting of at most n points.*

Proof. Let x and y be points of B and K any subset of A consisting of at most n points. If $T^{-1}(x)\cdot(A - K)$ separates[2] $T^{-1}(y)\cdot(A - K)$ in $A - K$, i.e., if $(A - K) - T^{-1}(x)\cdot(A - K) = A_1 + A_2$, $T^{-1}(y)\cdot(A - K)\cdot A_i \neq 0$ $(i = 1, 2)$, then this separation may be written in the form $(A - T^{-1}(x)) - K = A_1 + A_2$. Hence K separates $T^{-1}(y)$ in $A - T^{-1}(x)$, contrary to the definition of non-n-alternating. Thus the condition is necessary.

To establish the sufficiency, we notice that if there exist two points x, y in B and a cutting K of $A - T^{-1}(x)$ consisting of at most n points such that $T^{-1}(y)$ intersects both the sets A_1 and A_2 of the separation $A - (T^{-1}(x) + K) = A_1 + A_2$, then $(A - K) - T^{-1}(x)\cdot(A - K) = A_1 + A_2$ and therefore $T^{-1}(y)\cdot(A - K)$ is separated by $T^{-1}(x)\cdot(A - K)$ in $A - K$. Consequently, T is not non-alternating on $A - K$. This proves the sufficiency.

Lemma. *If $T(A) = B$ is non-n-alternating, B is non-degenerate, $y \in B$, and two points of $T^{-1}(y)$ are separated in A by a cutting K consisting of $k \leqq n + 1$ points, then $k = n + 1$ and $T(K) = y$.*

Proof. If $k \leqq n$, then T is non-alternating on the complement of K, by Theorem I. But this is impossible since $T^{-1}(y)$ intersects two components of this complementary set. Thus $k = n + 1$. If $T(K) \neq y$, there exists a point p in K such that $T(p) \neq y$. Then the set of n points $(K - p)$ separates $T^{-1}(y)$ in $A - T^{-1}(T(p))$, contrary to the fact that T is non-n-alternating. Therefore, $T(K) = y$.

One consequence of this lemma, namely, the fact that a point of order not

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[1] See G. T. Whyburn, *Non-alternating transformations*, American Journal of Mathematics, vol. 56 (1934), pp. 294–302.

[2] If L and M are subsets of N, we say that L "separates" M in N provided M is contained in $N - L$ and $N - L = N_1 + N_2$, where $N_1\bar{N}_2 = 0 = \bar{N}_1N_2$ and $MN_1 \neq 0 \neq MN_2$.