ON THE NECESSARY CONDITIONS FOR THE MINIMUM OF A DOUBLE INTEGRAL

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1. **Introduction.** The purpose of the present paper is to exhibit an intimate connection which exists between the theory of the calculus of variations for a double integral and the corresponding theory for a simple integral. The variation problem which will be discussed is that of minimizing an integral

$$(1.1) \qquad \int\int_A f(x, y, z_1, \cdots, z_n, z_{1x}, \cdots, z_{nx}, z_{1y}, \cdots, z_{ny})\, dx\, dy$$

in a certain class of sets of functions $z_i(x, y)$ $(i = 1, 2, \cdots, n)$ all of which take on the same values on the boundary of the region A of integration. When it is not assumed that the minimizing set $Z_i(x, y)$ $(i = 1, 2, \cdots, n)$ have continuous partial derivatives of order greater than the first, the differential equations which must be satisfied by the minimizing set were first derived (at least for the case $n = 1$) by Haar [1],[1] [2], who made use of his so-called Fundamental Lemma of the calculus of variations for double integrals. A survey of the literature concerning this Fundamental Lemma will be found in the Chicago dissertation of Miss Huke [3]; the proof of the lemma was simplified considerably by Haar in his last paper [4] on the subject.

Of the further necessary conditions for a minimum of the integral (1.1), the analogue for double integrals of the Legendre condition for the minimum of a simple integral was first established by Mason [5] for the case $n = 1$. The analogue of the Weierstrass condition was first proved by E. E. Levi [6] for the case $n = 1$ and by McShane [7] for the general case. We shall not be concerned in this paper with the analogues of the Jacobi condition.

The Fundamental Lemma of Haar is unnecessary for the development of the theory of the calculus of variations for the integral (1.1) and the well-known Du Bois-Reymond lemma of the theory for simple integrals suffices. Indeed, it will be shown below that there is associated with the problem for the integral (1.1) an auxiliary minimum problem for a simple integral and that the necessary conditions of Haar, Weierstrass, and Legendre for the problem involving the integral (1.1) can be respectively deduced as simple corollaries of the necessary conditions of Euler, Weierstrass, and Legendre for the auxiliary problem. The condition of Haar will be derived below in a modified form involving integral equations.

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.