REMARK ON THE THEOREM OF GREEN

By S. Bochner

The theorem of Green is an identity between an integral over a compact region R of an orientable locally Euclidean separable space S of class two[1] whose dimension will be denoted by n, and an integral over the boundary B of R, in case B is formed by "hypersurfaces":

$$\int_R \operatorname{div} \lambda dv = \int_B \lambda^i \Phi_i \, d\omega. \tag{1}$$

If R is closed, that is, compact and equal to S, thus having "no boundary", the integral over the boundary is to be put equal to zero:

$$\int_S \operatorname{div} \lambda dv = 0. \tag{2}$$

If R is contained in one coördinate neighborhood, the proof of (1) is comparatively simple, provided the boundary B is sufficiently smooth with respect to the coördinate system.[2] But the passage from the local case to a domain R in the large is rather laborious. It requires a cellular subdivision of R into sufficiently small subregions whose boundary is sufficiently smooth, an application of the local theorem to each subregion, and finally a justification of the mutual cancellation of the boundary terms arising from the artificial cellular partitions. Now this procedure is much too complicated and heavy in the case of formula (2) or in case of formula

$$\int_R \operatorname{div} \lambda dv = 0 \qquad (\lambda^i \equiv 0 \text{ on } B). \tag{3}$$

We want to show that these two formulas can be deduced in a much simpler fashion, even avoiding any complication that might be inherent to the local theorem itself. In particular we shall eliminate from the proof the concept of $[(n-1)$-dimensional$]$ volume on the boundary B.

Our space S being of class two, we can consider on it tensors of class one and tensor densities of class one. In particular, we assume the existence of a positive (non-vanishing) scalar density which, as in the special case of Riemann spaces, will be denoted by $\sqrt{g}$; in going over from coördinates $(x_1, \cdots, x_n)$ to coördinates $(y_1, \cdots, y_n)$, the quantity $\sqrt{g}$ is to be multiplied by the jacobian

Received December 19, 1936.

[1] That is, a space allowing local coördinate transformations with continuous first and second partial derivatives and a positive (non-vanishing) Jacobian.

[2] A. Duschek and W. Mayer, *Differentialgeometrie*, 1930, vol. II, p. 237