THE IMBEDDING OF RIEMANN SPACES IN THE LARGE

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1. The "classical" theory which describes the properties of an n-dimensional Riemann space immersed in an $(n + p)$-dimensional Euclidean space has its roots in the work of Voss and Ricci,[1] and is a natural generalization of the properties of a two-dimensional surface in ordinary three-space.

A more recent theory is that of W. Mayer, the latest refinement of which appeared in the Transactions of the American Mathematical Society in 1935 (vol. 38, pp. 267–309). We shall refer to this paper as "M". This theory combines the generalization of the properties of an ordinary surface with the generalization of those of a curve in an n-dimensional Euclidean space. Thus the theory speaks of the "first normal space" of an n-dimensional Riemann space as an analogue of the first ("principal") normal of a curve. Second and higher normal spaces also occur as extensions of the notions of the second and higher order normals of a curve. Associated with each normal space is a fundamental form, the form for the r-th normal space being of the $2(r + 1)$-th degree. These forms give a complete description of the geometry of the normal spaces. Mayer has shown that the coefficients of these forms are not entirely independent, but satisfy certain relations. The first object of this paper is to prove some additional relations of this nature and to determine a set of them which is the necessary and sufficient condition on a number of forms that they describe a Riemann space which is actually imbedded in a Euclidean space. These results are stated in Theorem VII. In the course of the argument (§5) we prove a new set of identities in the curvature tensors of order higher than the first which are analogous to those for the classical Riemann tensor.

The previous treatment of Mayer's theory had the further defect that it was purely local, referring to an unspecified domain about a suitable point. In this paper the differential equations which occur are investigated with the purpose of finding the maximum region within which they may be integrated. We define singular points and show that the results of Theorem VII hold for any simply connected portion of the space which does not contain a singular point.

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[1] A. Voss, *Zur Theorie der Transformation quadratischer Differentialausdrücke und der Krümmung höherer Mannigfaltigkeiten*, Math. Ann., vol. 16 (1880), pp. 129–179.

G. Ricci, *Formole fondamentali nella teoria generale di varietà e della loro curvatura*, Rend. dei Lincei, vol. 11 (1902), pp. 355–362.

A modern exposition of the theory is given by Eisenhart, *Riemannian Geometry*, 1925, Chapter IV.