THE STRUCTURE OF CERTAIN RATIONAL INFINITE ALGEBRAS

By Otto F. G. Schilling

G. Köthe has investigated infinite algebras over abstract fields in his paper "Ueber Schiefkörper unendlichen Ranges".[1] In this note we shall apply some of his fundamental results to infinite algebras over a finite algebraic number field. Application of the theory of finite algebras over algebraic number fields enables us to give explicit representations of the algebras considered as generalized crossed products.[2] Furthermore, we shall investigate the arithmetic in such infinite algebras.

1. **Algebraic theory.** Let k be an algebraic number field of finite degree over the field of all rational numbers. We consider algebras of infinite rank over k as centrum. We assume that the algebras A are countable, i.e., that there exists a countable set $\{a_1, a_2, \cdots, a_i, \cdots\}$ of elements of A, such that each element a of A can be represented as a finite sum $\sum_{\nu=1}^{\nu_0} k_{i_\nu} a_{i_\nu}$ with coefficients k_{i_ν} in the field k. Furthermore, we restrict ourselves to completely normal algebras which we define as follows.

Definition. A countable infinite algebra A with centrum k is called totally normal over k if every finite system $\{b_1, \cdots, b_m\}$ of elements of A lies in a normal simple finite algebra over k.

With slight modifications of Köthe's proofs one proves

Theorem 1. *For every totally normal algebra A there exists at least one defining sequence $k \subsetneq \cdots \subsetneq A_{i-1} \subsetneq A_i \subsetneq \cdots$ of normal simple algebras A_i.*

Conversely, we have

Theorem 2. *Every sequence $k \subsetneq \cdots \subsetneq A_{i-1} \subsetneq A_i \subsetneq \cdots$ of normal simple algebras A_i over k defines an infinite totally normal algebra A with the center k.*

Theorem 3. *Every totally normal algebra A over k is representable as the direct product of a countable infinity of normal simple algebras A_i of finite degrees over k. Such a decomposition is not necessarily uniquely determined.*

If we collect all simple normal systems of such a decomposition which belong to a fixed prime q, we have

Theorem 4. *Every totally normal algebra A over k is the direct product of a countable infinity of simple algebras $A^{(q)}$ which are primary with respect to k. The factors are uniquely determined except for isomorphisms.*

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[1] G. Köthe, Math. Annalen, vol. 105 (1931), pp. 15–39. See this paper also for the different notions used later on.

[2] For the theory of crossed products see the report of Max Deuring, *Algebren*, Berlin, 1935.