CERTAIN TERNARY CUBIC ARITHMETICAL FORMS

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1. By an obvious oversight, G. B. Mathews stated (without restriction on the integer n) that if the integer m is representable by integers x, y, z in the form

$$x^3 + ny^3 + n^2z^3 - 3nxyz,$$

it can be represented in an infinity of ways.[1] If n is the cube of a rational integer, and $m \neq 0$, the number of representations, if any, is finite. We shall show how these representations may be found. In certain simple cases (§5) it is possible to find the exact number of representations; in all cases we give an upper bound (§4) for this number.

2. We consider the representations by integers x, y, z of the integer m in the form

$$x^3 + t^3y^3 + t^6z^3 - 3t^3xyz, \tag{1}$$

where t is a constant integer $\neq 0$. If $m = 0$, we have the infinity of representations $(x, y, z) = (ht^2, ht, h)$, where h is an arbitrary integer (and possibly further representations).

Henceforth we shall take $m \neq 0$. Let $m = d\delta$, where d, δ are integers. From (1) we have

$$(x + ty + t^2z)(x^2 + t^2y^2 + t^4z^2 - txy - t^2xz - t^3yz) = d\delta;$$

hence we may take

$$x + ty + t^2z = d, \tag{2}$$

and equate the second factor to δ. Replacing t^2z by $d - x - ty$, we get

$$3(x^2 + txy + t^2y^2 - dx - dty) + d^2 - \delta = 0.$$

Thus

$$d^2 - \delta \equiv 0 \bmod 3; \qquad d^2 - \delta = 3h, \tag{3}$$

where h is an integer, and

$$x^2 + (ty - d)x + t^2y^2 - dty + h = 0. \tag{4}$$

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[1] G. B. Mathews, Proceedings of the London Mathematical Society, vol. 21 (1891), pp. 280–7; see also Dickson's *History of the Theory of Numbers*, vol. 2, 1920, p. 594. In applying Dirichlet's theory of representations by norms of algebraic integers, Mathews omitted to state, loc. cit., p. 281 (as he evidently intended) that the real cube root of the n he is considering must be irrational.