FUNCTIONS ARISING FROM DIFFERENTIAL EQUATIONS AND SERVING TO GENERALIZE A THEOREM OF LANDAU AND CARATHÉODORY

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Introduction. The hypergeometric linear differential equation which has for solutions the quarter periods of elliptic functions has been studied extensively by Fuchs[1] and Tannery.[2] By the use of a particular quotient of two of its solutions Picard and Landau[3] proved their remarkable theorems on analytic functions. If a certain transformation is made so that the exponents at the singular points $(0, 1, \infty)$ of this hypergeometric equation are all equal to each other, the equation so obtained is invariant with respect to the linear fractional dihedral group of order six, generated by $z' = 1 - z$ and $z' = 1/z$, where z is a complex variable.

The cyclic, dihedral, tetrahedral, octahedral and icosahedral groups are the only groups of finite order which are representable on linear fractional substitutions of a complex variable.[4] We shall specialize the exponent differences at the three singular points of the hypergeometric equation and then make the substitution $z = x^n$ on the independent variable. For each of the four specializations to be made we shall obtain an equation which is invariant with respect to some one of the first four groups named above, which is such that the exponent difference at each singular point is zero, and which has the property that an appropriate quotient function of two of its solutions has properties quite similar to those of the quotient function already mentioned.

We shall thus obtain four quotient functions, and by their use we shall obtain specific formulas for the radius of the circle in which every function of the form $F(x) = a_0 + a_1x + a_2x^2 + \cdots \ (a_1 \neq 0)$ must either have a singularity or assume one of a certain set of values as, for example, the n n-th roots of unity. Moreover, these radii will depend only on a_0, a_1, and this set of values.

1. **Specializations of the hypergeometric equation.** In the hypergeometric equation

$$(1) \quad 4z(z-1)v'' + 4\{(z-1)(1-\lambda) + z(1-\mu)\}v' + \{(1-\lambda-\mu)^2 - \nu^2\}\, v = 0$$

the singular points are at $z = 0, 1,$ and ∞ with exponents $0, \lambda$; $0, \mu$; $\frac{1}{2}(1-\lambda-\mu-\nu), \frac{1}{2}(1-\lambda-\mu+\nu)$, respectively.

Received August 26, 1935.

[1] L. Fuchs, Journal für Mathematik, vol. 71 (1870), pp. 91–127.

[2] J. Tannery, Annales de l'Ecole Normale, (2), vol. 15 (1879), pp. 169–194.

[3] E. Landau, Vierteljahrschrift der Natur. Gesellschaft, vol. 51 (1906), pp. 252–318.

[4] F. Klein, *Lectures on the Icosahedron*, p. 126.