FOURIER SERIES CONVERGENCE CRITERIA, AS APPLIED TO CONTINUOUS FUNCTIONS

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Whether or not there exists a continuous function whose Fourier series diverges everywhere, or almost everywhere, or on a set of points of positive measure, remains an unsolved problem. If a local condition is known which is sufficient to insure convergence of the Fourier series at a point, one is naturally led to raise the same question about the local condition itself: do there exist continuous functions which violate it at every point? For the criterion of Jordan, for example, the answer is clearly yes; for the more recent and more delicate criteria the question presents greater difficulty. Mazurkiewicz[1] and Kaczmarz[2] have shown that the answer is also affirmative in the case of the Dini criterion. It is the purpose of this note to answer this question for several more general convergence criteria.

Given any continuous function $f(x)$, which is periodic with period 2π, we define

$$\varphi(f; x; t) = \varphi(t) = f(x + t) + f(x - t) - 2f(x)$$

and

$$\Delta_\delta^k \varphi(t) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \varphi(t + j\delta) .$$

We first consider the condition

$$(L_k) \qquad \lim_{\delta \to +0} \int_\delta^\pi \frac{1}{t} \mid \Delta_\delta^k \varphi(t) \mid dt = 0$$

for a fixed integer k; any condition L_k insures convergence of the Fourier series, and the conditions are increasingly general; that is, L_k implies L_{k+1}. L_1 is the familiar Lebesgue criterion.

Let C be the space of continuous functions, periodic (2π), with the customary norm. We first prove

THEOREM 1. *For any positive integer k, the subset $A \subset C$ of functions such that for each x we have*

$$\overline{\lim_{\delta \to +0}} \int_\delta^\pi \frac{1}{t} \mid \Delta_\delta^k \varphi(t) \mid dt = +\infty$$

is of the second category in C, and its complement is of the first category.

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[1] Studia Mathematica, vol. 3 (1931), p. 114.

[2] Ibid., p. 189.