A FUNCTION NOT CONSTANT ON A CONNECTED SET OF CRITICAL POINTS

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1. **Introduction.** Let $f(x_1, \cdots, x_n)$ be a function of class C^m (i.e., with continuous partial derivatives through the mth order) in a region R. Any point at which all its first partial derivatives vanish is called a *critical point* of f. Suppose every point of a connected set A of points in R is a critical point. It is natural to suspect then that f is a constant on A. But this need not be so. We construct below an example with $n = 2$, $m = 1$, $A =$ an arc. The example may be extended to the case $n = n$, $m = n - 1$, $A =$ an arc. The arc and the function on the arc are easily defined. The extension of the function through the rest of the plane or space is given by a theorem of the author.[1]

The question settled in this paper was raised implicitly in a paper of W. M. Whyburn.[2] It is brought up by his definition of critical sets as the maximal connected subsets of the set of critical points on which the function takes a single critical value. Theorem 2 of Whyburn's paper shows that an example of the type given in the present paper can be constructed only by using critical sets which have points that cannot be joined in these sets by rectifiable arcs. It would be interesting to discover how far from rectifiable a closed set must be to be a set of critical points of some function but not a critical set of the function. It may be remarked that any closed set may be a critical set.[3]

For fixed n and m large enough, $m \geqq [(n-3)^2/16 + n]$, where $[n]$ is the integral part of n, f must be constant on any connected critical set, as shown by M. Morse and A. Sard in an unpublished paper.

The example shows that it is in general impossible to express the values of a function $f(x_1, \cdots, x_n)$ along a curve which is not rectifiable by means of an integral of a function of partial derivatives of f of order $\leqq n - 1$ along the curve.[4]

2. **The arc.** Let Q be a square of side 1 in the plane. Let Q_0, Q_1, Q_2, Q_3 be squares of side 1/3 lying interior to Q in cyclical order, each a distance 1/12

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[1] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 63–89, Lemma 2. We refer to this paper as AE.

[2] W. M. Whyburn, Bull. Amer. Math. Soc., vol. 35 (1929), pp. 701–708.

[3] See A. Ostrowski, Bull. des Sciences Math., Feb. (1934), pp. 64–72.

[4] For such an expression (using partial derivatives of any desired order) along a rectifiable curve, see H. Whitney, *Functions differentiable on the boundaries of regions*, Annals of Mathematics, vol. 35 (1934), pp. 482–485, (1) and (3).