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NEW COMPLEX- AND QUATERNION-HYPERBOLIC REFLECTION GROUPS

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To my father, John Allcock, 1940–1991

1. Introduction. In this paper, we carry out complex and quaternionic analogues of some of Vinberg's extensive study of reflection groups on real hyperbolic space. In [25] and [26], Vinberg investigated the symmetry groups of the integral quadratic forms diag$[-1, +1, \ldots, +1]$ or, equivalently, the Lorentzian lattices $I_{n,1}$. He was able to describe these groups very concretely for $n \leq 17$, and extensions of his work by Borcherds [7] and with Kaplinskaja [27] provide similar descriptions for all $n \leq 23$. In particular, the subgroup of $\operatorname{Aut} I_{n,1}$ generated by reflections has finite index just when $n \leq 19$.

In this paper, we study the symmetry groups of Lorentzian lattices over the rings $\mathscr{G}$ and $\mathscr{E}$ of Gaussian and Eisenstein integers and the ring $\mathscr{H}$ of Hurwitz integers (a discrete subring of the skew field $\mathbb{H}$ of quaternions). Most of the paper is devoted to the most natural of such lattices, the self-dual ones. The symmetry groups of these lattices provide a large number of discrete groups generated by reflections and acting with finite-volume quotient on the hyperbolic spaces $\mathbb{C}H^n$ and $\mathbb{H}H^n$. We construct a total of 19 such groups, including groups acting on $\mathbb{C}H^7$ and $\mathbb{H}H^5$. At least one of our groups has been discovered before, in the work of Deligne and Mostow [18], Mostow [22], and Thurston [24], but our largest examples are new. To the author's knowledge, quaternion-hyperbolic reflection groups have not been studied before.

Our results and techniques have found important application in work by the author, Carlson, and Toledo on the moduli space of complex cubic surfaces [3], [4]. Namely, this space is isomorphic to the Satake compactification of the quotient of $\mathbb{C}H^4$ by one of the reflection groups studied here. Furthermore, the moduli space of "marked" cubic surfaces may be realized as the Satake compactification of the quotient of $\mathbb{C}H^4$ by a congruence subgroup, which is also a reflection group in its own right.

The techniques used by Vinberg and others for the real hyperbolic case rely heavily on the fact that if a discrete group G is generated by reflections of $\mathbb{R}H^n$, then the mirrors of the reflections of G chop $\mathbb{R}H^n$ into pieces, and each piece may be taken as a fundamental domain for G. Work with complex or quaternionic reflection groups is much more complicated since hyperplanes have real codimension 2 or 4, and so the mirrors fail to chop hyperbolic space into pieces. Our solution to this problem is to

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