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HECKE THEORY AND EQUIDISTRIBUTION FOR THE QUANTIZATION OF LINEAR MAPS OF THE TORUS

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1. Introduction

1.1. Background. One of the key issues of "Quantum Chaos" is the nature of the semiclassical limit of eigenstates of classically chaotic systems. When the classical system is given by the geodesic flow on a compact Riemannian manifold M (or rather, on its cotangent bundle), one can formulate the problem as follows: The quantum Hamiltonian is, in suitable units, represented by the positive Laplacian $-\Delta$ on M. To measure the distribution of its eigenstates, we start with a (smooth) classical observable, that is, a (smooth) function on the unit cotangent bundle S^*M; via some choice of quantization from symbols to pseudodifferential operators, we form its quantization $\mathrm{Op}(f)$. This is a zero-order pseudodifferential operator with principal symbol f. The expectation value of $\mathrm{Op}(f)$ in the eigenstate ψ is $\langle \mathrm{Op}(f)\psi, \psi\rangle$.

Let ψ_j be a sequence of normalized eigenfunctions: $\Delta\psi_j + \lambda_j\psi_j = 0$, $\int_M |\psi_j|^2 = 1$. The problem then is to understand the possible limits as $\lambda_j \to \infty$ of the distributions

$$f \in C^\infty(S^*M) \longmapsto \left\langle \mathrm{Op}(f)\psi_j, \psi_j\right\rangle. \tag{1.1}$$

In the case where the geodesic flow is chaotic, it is assumed that the eigenfunctions are random, for instance, in the sense that the expectation values converge as $\lambda_j \to \infty$ to the average of f with respect to Liouville measure on S^*M. The validity of this for almost all eigenmodes if the classical flow is ergodic (so a very weak notion of chaos!) is asserted by Schnirelman's theorem [21],[1] a fact sometimes referred to as quantum ergodicity. The case where there are no exceptional subsequences is called "quantum unique ergodicity" (QUE). Its validity seems to be a very difficult problem, which is to date unsolved in any case where the dynamics are truly chaotic (see, however, Marklof and Rudnick [16], where QUE is proved for an ergodic, though nonmixing, model case).

1.2. Cat maps. In order to shed some light on the validity of QUE, we look at a "toy model" of the situation—the quantization of linear hyperbolic automorphisms

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[1]See Zelditch [24] and de Verdiere [5] for proofs.