Vol. 102, No. 3 DUKE MATHEMATICAL JOURNAL

ON FAMILY RIGIDITY THEOREMS, I

KEFENG LIU AND XIAONAN MA

0. Introduction. Let M, B be two compact smooth manifolds, and let $\pi : M \to B$ be a submersion with compact fiber X. Assume that a compact Lie group G acts fiberwise on M, that is, the action preserves each fiber of π. Let P be a family of elliptic operators along the fiber X, commuting with the action of G. Then the family index of P is

$$\operatorname{Ind}(P) = \operatorname{Ker} P - \operatorname{Coker} P \in K_G(B). \tag{0.1}$$

Note that $\operatorname{Ind}(P)$ is a virtual G-representation. Let $\operatorname{ch}_g(\operatorname{Ind}(P))$ with $g \in G$ be the equivariant Chern character of $\operatorname{Ind}(P)$ evaluated at g.

In this paper, we first prove a family fixed-point formula that expresses $\operatorname{ch}_g(\operatorname{Ind}(P))$ in terms of the geometric data on the fixed points X^g of the fiber of π. Then by applying this formula, we generalize the Witten rigidity theorems and several vanishing theorems proved in [Liu3] for elliptic genera to the family case.

Let $G = S^1$. A family elliptic operator P is called rigid on the equivariant Chern character level with respect to this S^1-action, if $\operatorname{ch}_g(\operatorname{Ind}(P)) \in H^*(B)$ is independent of $g \in S^1$. When the base B is a point, we recover the classical rigidity and vanishing theorems. When B is a manifold, we get many nontrivial higher-order rigidity and vanishing theorems by taking the coefficients of certain expansion of ch_g. For the history of the Witten rigidity theorems, we refer the reader to [T], [BT], [K], [L2], [H], [Liu1], and [Liu4]. The family vanishing theorems that generalize those vanishing theorems in [Liu3], which in turn give us many higher-order vanishing theorems in the family case. In a forthcoming paper, we extend our results to general loop group representations and prove much more general family vanishing theorems that generalize the results in [Liu3]. We believe there should be some applications of our results to topology and geometry, which we hope to report on a later occasion.

This paper is organized as follows. In Section 1, we prove the equivariant family index theorem. In Section 2, we prove the family rigidity theorem. In the last part of Section 2, motivated by the family rigidity theorem, we state a conjecture. In Section 3, we generalize the family rigidity theorem to the nonzero anomaly case. As corollaries, we derive several vanishing theorems.

Acknowledgements. We would like to thank W. Zhang for many interesting discussions. Part of this work was done while the second author was visiting the Institut des Hautes Études Scientifiques. He would like to thank Professor J. P. Bourguignon and IHES for their hospitality.

Received 23 April 1999.
1991 *Mathematics Subject Classification*. Primary 14D05; Secondary 57R91.