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SYMPLECTIC MODULAR SYMBOLS

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1. Introduction

1.1. Let G be a semisimple algebraic group defined over $\mathbb{Q}$ of $\mathbb{Q}$-rank ℓ, and let X be the associated symmetric space. Let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Then $H^*(\Gamma;\mathbb{Z}) = H^*(\Gamma\backslash X;\mathbb{Z})$, and this cohomology vanishes for $* > N$, where $N = \dim(X) - \ell$, the cohomological dimension of Γ.

The theory of modular symbols as formulated by Ash [2] constructs an explicit spanning set for $H^N(\Gamma;\mathbb{Z})$ as follows. Let $\mathcal{B}$ be the Tits building associated to G [17]. By the Solomon-Tits theorem, $\mathcal{B}$ has the homotopy type of a wedge of $(\ell-1)$-spheres, and thus $\tilde{H}_*(\mathcal{B};\mathbb{Z})$ is nonzero only in dimension $\ell-1$. Using the Borel-Serre compactification of the locally symmetric space $\Gamma\backslash X$, we may construct a map

$$\Phi : H_{\ell-1}(\mathcal{B};\mathbb{Z}) \longrightarrow H^N(\Gamma;\mathbb{Z}) \tag{1}$$

that is surjective (cf. §2). Because the left-hand side of (1) is generated by fundamental classes of apartments of $\mathcal{B}$, this provides a geometric spanning set for $H^N(\Gamma)$. These cohomology classes (or rather, their duals in homology) are called *modular symbols*.

1.2. The modular symbols provide a spanning set for $H^N(\Gamma;\mathbb{Z})$, but they do not provide a finite spanning set, a distinction that is important for applications. However, suppose $K/\mathbb{Q}$ is a number field with euclidean ring of integers $\mathcal{O}$, and let $G(\mathbb{Q}) = \mathrm{SL}_n(K)$ and $\Gamma \subset \mathrm{SL}_n(\mathcal{O})$. Then in [6], Ash and Rudolph determine an explicit finite spanning set—the *unimodular symbols*—and present an algorithm to write a modular symbol as a sum of unimodular symbols (cf. §2.9). This algorithm, in conjunction with certain explicit cell complexes, can be used to compute the action of the Hecke operators on $H^N(\Gamma)$. In turn, through work of Ash, Pinch, and Taylor [5], Ash and McConnell [4], and van Geemen and Top [18], corroborative evidence has been produced for certain aspects of the "Langlands philosophy." In particular, in the case of $\Gamma \subset \mathrm{SL}_3(\mathbb{Z})$, many examples of representations of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ have been found that appear to be associated to cohomology classes of Γ.

1.3. In this paper we solve the finiteness problem for the symplectic group: $G(\mathbb{Q}) = \mathrm{Sp}_{2n}(K)$ and Γ of finite index in $\mathrm{Sp}_{2n}(\mathcal{O})$, where $\mathcal{O}$ is euclidean. We characterize a finite spanning set of $H^N(\Gamma;\mathbb{Z})$ and present an algorithm (Theorem 4.11)

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