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QUASICONFORMALITY, QUASISYMMETRY, AND REMOVABILITY IN LOEWNER SPACES

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APPENDIX BY JUSSI VÄISÄLÄ

To Professor Hans Martin Reimann

1. Introduction. Let X and Y be metric spaces and $f : X \to Y$ a homeomorphism. Then the distortion of f at a point $x \in X$ is

$$H(x) := \limsup_{r\to 0} \frac{L(x,r)}{l(x,r)}, \tag{1}$$

where

$$L(x,r) := \sup\left\{|f(x)-f(y)| : |x-y| \le r\right\},$$
$$l(x,r) := \inf\left\{|f(x)-f(y)| : |x-y| \ge r\right\},$$

and by $|x-y|$ we denote the distance between x and y in a metric space. We say that f is quasiconformal if there is a constant H so that $H(x) \le H$ for every $x \in X$. This infinitesimal condition is easy to state but not easy to use. For instance, it is not clear from the definition if the inverse mapping is quasiconformal as well. It was recently shown by Heinonen and Koskela [7] that, for a large class of metric spaces, quasiconformal mappings satisfy a stronger, global condition:

$$H(x,r) := \frac{L(x,r)}{l(x,r)} \le H' < \infty$$

for all $x \in X$ and all $r > 0$. This holds under the conditions that $X = Y$ is an Ahlfors-David Q-regular Loewner space with $Q > 1$ and that f maps bounded sets to bounded sets. Let us call maps that satisfy the above global inequality quasisymmetric. This condition is weaker than the usual definition of quasisymmetry but results in the same class of mappings for a large class of metric spaces including the ones considered in Theorems 1.1, 1.2, and 1.3; see also [7]. Here the (Ahlfors-David) Q-regularity of the metric space X means that X is equipped with a Borel measure μ and there is a constant $C_\mu \ge 1$ such that

$$C_\mu^{-1} r^Q \le \mu\big(B(r)\big) \le C_\mu r^Q,$$

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