J. Math. Kyoto Univ. (JMKYAZ)
14-2 (1974) 391-411

On the cut locus and the topology of Riemannian manifolds

By

Kunio SUGAHARA

(Received Sep. 19, 1973)

1. Introduction.

Let M be a connected complete Riemannian manifold with $\dim M \geqq 2$. Let p be a point in M and let $Q(p)$ (resp. $C(p)$) be the conjugate locus (resp. the cut locus) in the tangent space $T_p(M)$ to M at p. (For the precise definitions of $Q(p)$ and $C(p)$, see section 2.) We say that M satisfies condition (C) at p or the pair (M, p) satisfies condition (C) if $Q(p)$ and $C(p)$ do not have common points.

In this paper, we study the structure of the cut locus $C(p)$ and the topology of the Riemannian manifold M assuming that M satisfies condition (C) at a given point p.

A. D. Weinstein [8] showed that any compact manifold M with $\dim M \geqq 3$ always admits a Riemannian metric g which satisfies condition (C) at some point p in M. Therefore, for our purpose, we need some further assumptions on the Riemannian manifold. The principal tool in our study is the map $N_p: C(p) \rightarrow \mathbf{N} \cup \{+\infty\}$ defined by

$$N_p(v) = \#\{w \in C(p);\ \exp_p v = \exp_p w\}$$

for all $v \in C(p)$, where $\exp_p: T_p(M) \rightarrow M$ denotes the exponential map.

The main results are stated as follows.

Theorem A. *Assume that (M, p) satisfies condition (C). Then we have*

(1) *The set $N_p^{-1}(2) = \{v \in C(p);\ N_p(v) = 2\}$ is open and dense in*