J. Math. Kyoto Univ. (JMKYAZ)
20-4 (1980) 591-623

Analytic mappings of a Riemann surface of finite type into a torus

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(Received March 30, 1979)

Introduction

The study on the existence of analytic mappings of a Riemann surface into another generally involves much difficulty. Obstruction lies, above all, in the non-planer character of the image surface. The purpose of the present paper is to investigate analytic mappings of what we call a Riemann surface of finite type into a torus. Namely, the domain surface R_N is a closed Riemann surface with a finite number ($=N$) of points (punctures) removed. The image surface, on the other hand, is a closed surface T of genus one. In such a case, we can make use of the finite complex plane which is the universal covering surface of T.

After preparing some fundamental facts, we shall first prove a theorem (Theorem 1) which gives a necessary and sufficient condition for the existence of an analytic mapping f of R_N into T with two prescribed properties; one is purely topological and the other is purely analytic. The topological condition imposed on f is the assignment of the homomorphism between the first homology groups which is to be induced by f, and the analytic condition is the pre-assignment of the behavior of f near the punctures (which are the isolated singular points of f). Theorem 1 is proved by means of real normalization of periods of Abelian differentials, while we shall later make use of complex normalization to prove a corresponding theorem (Theorem 5). These two results, Theorems 1 and 5, are thus the same in essence. Each of them has, however, an advantage over the other in applications. Compare Theorem 6 with Theorem 7.

Historically, such a problem was first considerd for closed surfaces ($N=0$). The existence and the determination of explicit form of the mapping f were mainly studied. See Krazer [8], the last chapter. We shall also recall some relevant known facts: For any homomorphism between the first homology groups of a closed surface of positive genus and a torus, there always exists a *continuous* mapping which induces the homomorphism (H. Hopf [7]). An *analytic* mapping, however, does not necessarily exist (Gerstenhaber [4]). On the contrary, if the domain surface has N punctures, $N\geqq 1$, then every homomorphism between the homology groups is induced by an analytic mapping of R_N into T. In