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On the conjugacy of nilpotent elements in the classical Lie algebras in relation to their representations

By

Tôru Umeda

Introduction.

Let G be an algebraic group over an algebraically closed field K, and $\mathfrak{g}$ the Lie algebra of G. The subject of the present paper is on the conjugacy of nilpotent elements in $\mathfrak{g}$ in relation to representations of G. Let us explain our problems. Let (σ, V) be a finite dimensional rational representation of G. We use the same notation for the corresponding representation of $\mathfrak{g}$. For nilpotent elements X, X' in $\mathfrak{g}$, we consider the following two types of conjugacy:

$$(C) \qquad X \sim X' \qquad \text{under} \quad \mathrm{Ad}(G)\,,$$

$$(C_\sigma) \qquad \sigma(X) \sim \sigma(X') \qquad \text{under} \quad GL(V)\,.$$

It is easy to see that (C) implies (C_σ). Then there arise naturally the converse problems:

Problem I $(G; \sigma)$. Does it hold that (C_σ) implies (C)?

And more generally,

Problem $\tilde{\mathrm{I}}$ $(G; \sigma)$. Find all the conjugacy classes in the sense (C) which are glued up in (C_σ).

In the present paper, we are mainly interested in a special case where σ is the adjoint representation σ_1. We call these problems Problem $\mathrm{I}_1(G)$, Problem $\tilde{\mathrm{I}}_1(G)$, for this special case.

For the case where $\mathfrak{g}$ is the simple Lie algebra of exceptional type E_6, E_7, E_8, T. Hirai announced in [3, Th. 6] the affirmative answer to Problem $\mathrm{I}_1(G)$ when the characteristic of K is good for $\mathfrak{g}$. This result is the first motive for our study, and we are concerned with the classical cases.

Let us specify our object and state the main results. In this paper K is always assumed to be of characteristic zero. To Problem $\mathrm{I}_1(G)$ with $G = GL(n, K)$, $Sp(n, K)$, $O(n, K)$, we obtain the following answers both in the affirmative and the negative.

Theorem A (*Corollary to Theorem* 1). *Let G be one of the groups $GL(n, K)$,*

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