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Generalized divisors on Gorenstein curves and a theorem of Noether

By

Robin HARTSHORNE

§0. Introduction.

Recently while considering the possible special linear systems which can exist on a nonsingular plane curve, we rediscovered an old result of Max Noether. The problem is to find the largest possible dimension of a linear system of degree d on a plane curve of degree k. The answer is that the linear systems of maximal dimension are the ones which "exist naturally" on the curve because of its plane embedding, namely the linear systems cut out by all plane curves of some other fixed degree, plus a few extra points or minus a few assigned base points. (See (2.1) for the exact statement.)

A natural way to approach the problem is by induction on k. If the divisor D on the curve C is nonspecial, then $h^0(\mathcal{O}_C(D))$ can be found by the Riemann-Roch theorem. If on the other hand D is special, then it is contained in a canonical divisor. The canonical divisors are cut out on C by curves of degree $k-3$ in the plane, so D is contained in a curve of degree $k-3$, and one can try to use induction. The trouble is that the new curve of lower degree containing D may not be nonsingular. For this reason we have developed a theory of generalized divisors on Gorenstein curves, which appears in §1. We believe this theory may be useful in other contexts, and by way of example have given a new proof of a theorem of Fujita (1.6) telling when the canonical divisor on a Gorenstein curve is very ample. This result should simplify the beginning theory of divisors on K3 surfaces as given in the paper os Saint-Donat [19].

Since any plane curve is Gorenstein, we can use the theory of §1, combined with Bertini's theorem, to formulate and prove Noether's theorem for generalized divisors on irreducible plane curves. This is done in §2. It turns out that Noether's original proof followed the same method, but we must consider that it is incomplete, because he assumes without justification that a curve of lower degree containing the divisor D can be chosen to be nonsingular. Meanwhile another proof of Noether's theorem, for nonsingular plane curves, has been given by Ciliberto [4] using a different method.

An important application of Noether's theorem, and indeed the reason for

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