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A note on the coefficients of Hilbert polynomial

By

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Let (Q, m) be a local ring of dimension d and q an m-primary ideal. It is well known that for sufficiently large values of n (notation: $n \gg 0$) $l_Q(Q/q^{n+1})$, the length of the Q-module Q/q^{n+1} is a uniquely determined polynomial of degree d, called the Hilbert polynomial and is given by

$$l_Q(Q/q^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

where the integers $e_0, e_1, \dots, e_d$ depend on q and are known as normalized Hilbert coefficients. e_i is sometimes written as $e_i(q)$ to emphasize its dependence on q.

It is easily seen that e_0 is positive. In case Q is a Cohen-Macaulay ring, Northcott [2] showed that e_1 is non-negative and Narita [1] showed that in this case e_2 is non-negative as well. Narita gave an example of a Cohen-Macaulay ring and an m-primary ideal q such that $e_3(q)$ is negative.

The purpose of this note is to show that for any integer $d \geqslant 3$, it is possible to construct an example of a Cohen-Macaulay ring (Q, m) of dimension d and an m-primary ideal q such that $e_d(q)$ is negative.

We use the arguements given in [1] to obtain explicit values of the normalized Hilbert coefficients and use them subsequently to test our examples for the claim made above. To give a general treatment we must introduce some auxilliary notations which are explained at the appropriate place. Throughout this note (Q, m) denotes a Cohen-Macaulay ring with infinite residue field.

§1:

To start with we quote the following result

1.1 (Northcott [2]). *Let* $\dim Q = d$ *and* w *a superfical element of* q. *Suppose that* w *is not a zero divisor. Let* $\bar{Q} = Q/Qw$ *and* $\bar{q} = q/Qw$. *Then we have*

(i) $l_{\bar{Q}}(\bar{Q}/\bar{q}^{n+1}) = l_Q(q^{n+1} : w/q^{n+1})$ *for all* $n > 0$

(ii) $l_{\bar{Q}}(\bar{Q}/\bar{q}^{n+1}) = l_Q(Q/q^{n+1}) - l_Q(Q/q^n)$ *for all* $n \gg 0$

(iii) $e_i(q) = e_i(\bar{q}), \quad 0 \leq i \leq d-1$.

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