J. Math. Kyoto Univ. (JMKYAZ)
32-3 (1992) 621-654

Sobolev spaces on a Riemannian manifold and their equivalence

By

Nobuo YOSHIDA

1. Introduction

Analysis of Laplacian on Riemannian manifolds has been studied in a large number of literatures and many classical results on the Euclidean space $\mathbf{R}^n$ are carried over to the setting of a Riemannian manifold. The problem we discuss here is the equivalence of Sobolev spaces on the manifold. In the case of $\mathbf{R}^n$, a classical result asserts that there exist positive constants $c = c_{p,k}$ and $C = C_{p,k}$ for every $p \in (1, \infty)$ and $k = 1, 2, \dots$ such that for all $u \in C_c^\infty(\mathbf{R}^n)$ we have

$$c \sum_{j=0}^{k} \|(\nabla)^j u\|_p \leq \|(1 - \Delta)^{k/2} u\|_p \leq C \sum_{j=0}^{k} \|(\nabla)^j u\|_p . \tag{1.1}$$

Here, Δ is the Laplacian and ∇ is the gradient ([7], [18]). This result implies the equivalence of Sobolev spaces with corresponding norms and a proof of (1.1) is given by the Littlewood-Paley-Stein inequality ([18]). In the late seventies, P. A. Meyer developed a probabilistic approach to the Littlewood-Paley-Stein theory ([16]). Recently, D. Barky applied this idea to prove a beautiful result corresponding to (1.1) for $u \in C_c^\infty(M)$ in the case of $k = 1$, where M is a complete Riemannian manifold with Ricci curvature bounded from below ([5]). Some analogous results for differential forms and tensor fields are also obtained in [5]. But as for tensor fields, the method there requires a rather strong hypothesis that $\nabla \operatorname{Ric} = 0$ and even for $u \in C_c^\infty(M)$, (1.1) for $k \geq 2$ does not follow directly from the theorems in [5]. These difficulties are mainly caused by the non-commutativity of ∇ and Δ on the manifold. To overcome this problem, we consider a further generalization of the Littlewood-Paley-Stein inequality. This enables us to prove (1.1) for any integer $k \geq 1$, and for a section u of a vector bundle over the manifold, which generalizes and improves the results in [5].

Before stating our results we explain our setting. Let (M, g) be a complete Riemannian manifold and F be a vector bundle over M with an inner product $(\cdot|\cdot)$ and a connection ∇. We assume that $(\cdot|\cdot)$ is parallel in the sense that $\partial|u|^2/\partial x^i = 2(\nabla_i u|u)$ for all $u \in \Gamma(F)$, where $\Gamma(F)$ is the space of C^∞-sections, $|u|^2 = (u|u)$ and $\nabla_i = \nabla_{\partial/\partial x^i}$ is the covariant derivative with respect to ∇ (As usual, the summation signs for repeated indices are ommited). In the sequel, we often

Received November 8, 1990