J. Math. Kyoto Univ. (JMKYAZ)
33-1 (1993) 261-283

Spectral flow and intersection number

By

Kenro FURUTANI and Nobukazu OTSUKI

Dedicated to Professor Nobuhiko Tatsuuma on his 60-th birthday

§ 1. Introduction

Let $\mathscr{F} = \mathscr{F}(H)$ be the set of bounded Fredholm operators on a separable (complex) Hilbert space H of infinite dimension. $\mathscr{F}$ is a classifying space for the complex K-group. A subset $\hat{\mathscr{F}}$ of $\mathscr{F}$ consisting of selfadjoint operators has three components:

$$\hat{\mathscr{F}} = \hat{\mathscr{F}}_+ \cup \hat{\mathscr{F}}_- \cup \hat{\mathscr{F}}_* . \tag{1-1}$$

$\hat{\mathscr{F}}_+$ ($\hat{\mathscr{F}}_-$) consists of essentially positive (negative) operators and $\hat{\mathscr{F}}_*$ consists of others. $\hat{\mathscr{F}}_\pm$ are contractible and $\hat{\mathscr{F}}_*$ is a classifying space for K^{-1}-group ([AS]). Especially we have

$$\pi_1(\hat{\mathscr{F}}_*) \cong \mathbf{Z} . \tag{1-2}$$

An isomorphism of (1-2) is given by, so called, the spectral flow. It is defined as the number of eigenvalues (with directions) that change signs when the parameter of a loop in $\hat{\mathscr{F}}_*$ goes around ([APS1, 2]). This definition is more clarified by considering a subspace $\hat{F}(\infty)$ of $\hat{\mathscr{F}}_*$, which has the same homotopy type with the whole space $\hat{\mathscr{F}}_*$ and has a spectrally nice property in a sense ([BW1]):

$$\hat{F}(\infty) = \{A \in \hat{\mathscr{F}}_*\colon \|A\| = 1, \ \text{the essential spectra } \sigma_{ess}(A) \text{ of } A \text{ are just } \{-1, 1\} \text{ and other spectra } \sigma(A)\backslash\sigma_{ess}(A) \text{ are the finite number of eigenvalues}\} . \tag{1-3}$$

Let $l\colon [0, 1] \to \hat{F}(\infty)$ be a continuous loop, then the graph of the spectrum of l can be parametrized through a finite monotone sequence of continuous functions:

$$\lambda_j\colon [0, 1] \to [-1, 1] \qquad j = 1, \ldots, N , \tag{1-4}$$

(N is the maximal number of the eigenvalues $\in \sigma(l(t))\backslash\sigma_{ess}(l(t))$ with multiplicities of the operator $l(t)$ $(0 \leq t \leq 1)$)
$-1 \leq \lambda_1(t) \leq \cdots \leq \lambda_N(t) \leq 1$

Communicated by Prof. T. Hirai, September 30, 1991, Revised December 12, 1991