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Failure of analytic hypoellipticity for some operators of $X^2 + Y^2$ type

Dedicated to Professor Tosinobu Muramatu on his 60th birthday

By

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0. Introduction and result

A differential operator P is said to be hypoelliptic, if for any C^∞ function f in some open set U all solutions u to $Pu = f$ belong to $C^\infty(U)$. Also P is said to be analytic hypoelliptic, if $f \in C^\omega(U)$ implies $u \in C^\omega(U)$. Let Ω be an open set in $\mathbf{R}^n$ and $X_1, \dots, X_r$ be real vector fields with analytic coefficients. It is well known that, if $X_1, \dots, X_r$ and their commutators $[X_{j_1}, X_{j_2}], \dots, [X_{j_1}[X_{j_2}, \dots, [X_{j_{k-1}}, X_{j_k}] \cdots] \dots$ generate the tangent space $T_x\mathbf{R}^n$ for all $x \in \Omega$ then the operator

$$P = \sum_{j=1}^{r} X_j^2 \tag{1}$$

is hypoelliptic in Ω (L. Hörmander [7]).

Note that such an assumption as above is not sufficient for analytic hypoellipticity (*cf.* F. Treves [12], D. S. Tartakoff [11] and A. Grigis-J. Sjöstrand [4]). Indeed, there are some negative results. Some hypoelliptic operators of type (1) were shown to be not analytic hypoelliptic. Such operators can be seen, for example, in the following papers: M. S. Baouendi- C. Goulaouic [1], G. Métivier [8], B. Helffer [6], Pham The Lai-D. Robert [9], N. Hanges-A. A. Himonas [5] and M. Christ [2]. The purpose of the present paper is to give new examples of hypoelliptic operators which fail to be analytic hypoelliptic.

Here, we consider the operator

$$P = \frac{\partial^2}{\partial x^2} + \left(x^k \frac{\partial}{\partial y} - x^l \frac{\partial}{\partial t} \right)^2 \tag{2}$$

in $\mathbf{R}^3$. If the non negative integers k, l satisfy $k < l$, then Hörmander's theorem can be applied, hence the operator P is hypoelliptic. With this hypothesis the result of the present paper is following

Theorem. *The operator P in* (2) *is not analytic hypoelliptic, if either of the following assumptions is satisfied*:

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