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Construction of the Green function on Riemannian manifold using harmonic coordinates

By

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0. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary. We denote the Levi-Civita Connection of (M, g) by ∇, and the Laplace operator by Δ. In this paper, we will prove an L^p-estimate for the Laplace operator:

$$\|\nabla^2 u\|_p \leq C\|\Delta u\|_p.$$

Naturally, the constant C depends on geometric data of (M, g). The main purpose of this paper is to estimate the constant C in terms of the diameter, the injectivity radius, and the lower bound of the Ricci tensor.

For the purpose of this, we construct the Green function using a parametrix. In [2, 3], Aubin used the Riemannian distance function $d(x, y)$ to construct a parametrix of the Green function. However, the second derivatives of the distance function cannot be estimated in terms of the Ricci tensor. In fact, we need a bound of Riemann curvature tensor in order to yeild an estimate of $\Delta d(x, y)$. (Here the Laplace operator Δ acts on $d(x, y)$ with respect to the argument y.) Therefore we construct a parametrix utilizing the harmonic coordinate of [1]. In the course of this we estimate the Green function and its first derivatives near the singularity in Section 6, and, using the estimate of the second derivative of the parametrix, we show the Calderon-Zygmund type inequalities in Section 6, from which we can easily obtain an L^p-estimate for the Laplace operator.

We denote the diameter by D, the injectivity radius by i_0, the volume by V, and the Ricci tensor by Ric. We fix a non-negative constant Λ for which the bound $\mathrm{Ric} \geq -(n-1)\Lambda g$ is satisfied.

For $x \in M$, the Green function G_x is a unique smooth functions on $M \setminus \{x\}$ that satisfies $\Delta G_x = \delta_x - V^{-1}$ as distributions and $\int_M G_x d\mu = 0$, where δ_x is the Dirac function at x and $d\mu$ is the Riemannian volume form.

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