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# Surfaces of general type whose canonical map is composed of a pencil of genus 3 with small invariants

By

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## 0. Introduction

Let $X$ be a minimal surface of general type over the complex number field. Assume that $p_g(X) \geq 3$, and $|K_X|$ is composed of a pencil. The existence of such surfaces was known as early as 1948 by Pompilij's examples. Later there have been studies by Beauville, Debarre, Xiao and others ([3], [5], [10], [12]). Refer to Section 2 of [4] for a nice survey.

Let $b$ denote the geometric genus of the image of the canonical map and let $g$ denote the genus of a general member of the pencil of which $|K_X|$ is composed. Assume that $g \geq 3$. Then the inequality

$$K_X^2 \geq 4p_g(X) + 4(b-1) \tag{1}$$

is valid with very few exceptions (cf. Theorem 2.3 of [4]).

In this paper we will give an example with $p_g = 3$, $b = 0$, $g = 3$ and $K^2 = 7$. Then we will prove that is the lowest possible $K^2$.

The other possible exception to (1) is the case $p_g = 4$ and $K_X^2 = 9$, which was proposed as an open problem in [11]. We will prove that this case does not occur, and consequently there is only one exception to (1).

## 1. Preliminaries

**1.1. $\mathbf{P}^2$-bumdles over $\mathbf{P}^1$.** First we state some basic facts about $\mathbf{P}^2$-bundles over the projective line $\mathbf{P}^1$, which will be used throughout this paper. We will use $\mathcal{O}(n)$ to denote either the invertible sheaf of degree $n$ on $\mathbf{P}^1$ or its corresponding line bundle, depending on the context.

Let $V$ be a vector bundle of rank 3 over $\mathbf{P}^1$. It is well-known that $V$ can be decomposed into a direct sum of line bundles, i.e., $V \cong \mathcal{O}(k) \oplus \mathcal{O}(m) \oplus \mathcal{O}(n)$. Let $W = \mathbf{P}(V)$ be the associated $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ and let $f: W \to \mathbf{P}^1$ denote the natural map. Since $\mathbf{P}(V \otimes L) \cong \mathbf{P}(V)$ for any line bundle $L$, we may

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