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Almost minimal embeddings of quotient singular points into rational surfaces

By

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0. Introduction

Let k be an algebraically closed field of characteristic zero. Let $\bar{X}$ be a normal algebraic surface with only one quotient singular point P. Let $f: X \rightarrow \bar{X}$ be a minimal resolution of $\bar{X}$ and let $D=\sum_{i=1}^{n} D_i$ be the reduced exceptional divisor with respect to f, where the D_i are irreducible components. Define a $\mathbf{Q}$-divisor $D^{\#}=\sum_{i=1}^{n} \alpha_i D_i$ such that $(D^{\#}+K_X \cdot D_i)=0$ for every $1 \leq i \leq n$. Since the intersection matrix of D is negative definite and $(D_i^2) \leq -2$, $D^{\#}$ is uniquely defined and $0 \leq \alpha_i < 1$. We say that a pair (X, D) is *almost minimal* if, for every irreducible curve C, $(D^{\#}+K_X \cdot C) \geq 0$ or the intersection matrix of $C+\mathrm{Bk}(D)$ is not negative definite, where $\mathrm{Bk}(D)=D-D^{\#}$ (see Miyanishi-Tsunoda [11, p. 226]). We say that the singular point P is *almost minimal* in $\bar{X}$ if the pair (X, D) is almost minimal. Then $D^{\#}+K_X \equiv f^*(K_{\bar{X}})$, and $\bar{X}$ is log relatively minimal (cf. Gurjar-Miyanishi [3]). By virtue of [11, 1.11], we can construct the almost minimal singular points from any quotient singular points which might be changed from the original singularities.

In the present article, we study such singularities. In the section 1, we study the case of the logarithmic Kodaira dimension $\bar{\kappa}(X-D)=-\infty$ using the Mori theory [13], and classify such singular points when $\mathrm{Supp}(D)$ is contained in a fiber of a certain $\mathbf{P}^1$-fibration (Theorem 1.1). In [17, Proposition 2.2], Tsunoda classified all the almost minimal quotient singular points on rational surfaces for which $\bar{\kappa}(X-D)=0$. In the section 2, we study and give a classification in the case where $\bar{\kappa}(X-D)=1$ and X is a rational surface (Theorem 2.5). In the section 3, we classify the almost minimal pair (X, D) where D is irreducible by using some results in Mohan Kumar-Murthy [14] and Iitaka [5]. Finally, in the section 4, we study the structure of (X, D) and give examples when $\bar{\kappa}(X-D)=0$ and X is a rational surface (Theorem 4.1).

The terminology is the same as the one in [11]. By a $(-n)$-curve we mean a nonsingular rational curve with self-intersection number $-n$. A reduced effective divisor D is called an SNC divisor (an NC divisor, resp.) if

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